EASTERN UNIVERSITY, SRI LANKA

SPECIAL DEGREE EXAMINATION IN MATHEMATICS

-(2003/2004), (JUL./AUG.' 2005)

Part II

MT 404 - ALGEBRAIC TOPOLOGY

Answer all questions

Time: Three hours

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1. (a) Define a G-space, and prove, if X is a G-space, then the canonical projection $\Pi : X \to X|_G$ is an open map. [20]

   (b) Establish that a subset S of a topological space X is compact if and only if S is compact as a space under the induced topology. [30]

   (c) "A compact subset A of a Hausdorff space is closed." Verify this satatement. [20]

   (d) Let Y be a quotient space of the topological space X determined by the surjective map $f : X \to Y$. If X is compact Hausdorff and f is closed, prove that Y is compact Hausdorff. [30]

2. (a) Is the interval $[a, b] \subseteq \mathbb{R}$ connected?

Prove your assertion.

[10]

(b) Show that, upto homeomorphism, S^1 is the only compact connected 1-manifold.

[30]

(c) Let C be a Jordan curve given by $f : S^1 \to \mathbb{R}^2$. Prove that, for every $\epsilon > 0$, there exists a Jordan polygon C' given by $f' : S^1 \to \mathbb{R}^2$ such that $|f(x) - f'(x)| < \epsilon$ for all $x \in S^1$.

[30]

(d) If C is a Jordan curve, prove that $\mathbb{R}^2 - C$ has at least two components.

[30]

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3. (a) Define a strong deformation retract of a space X.

If $X = Y - \{(2,0), (-2,0)\}$, where Y is the subset of $\mathbb{R}^2$ defined by $Y = \{x = (x_1, x_2) : (x_1 - 1)^2 + x_2^2 = 1\} \cup \{x = (x_1, x_2) : (x_1 + 1)^2 + x_2^2 = 1\}$, show that the subset $\{x_0\}$, where $x_0 = (0,0)$, of X is a strong deformation retract of X.

[25]

(b) Let $\phi, \psi : X \to Y$ be continuous mappings between topological spaces. Let $F : \phi \simeq \psi$ be a homotopy. If $f : I \to Y$ is the path from $\phi(x_0)$ to $\psi(x_0)$ given by $f(t) = F(x_0, t)$, show that the homomorphisms $\phi_* : \Pi(X, x_0) \to \Pi(Y, \phi(x_0))$ and $\psi_* : \Pi(X, x_0) \to \Pi(Y, \psi(x_0))$ are related by $\psi_* = U_f \phi_*$, where U_f is the isomorphism from $\Pi(Y, \phi(x_0))$ to $\Pi(Y, \psi(x_0))$ determined by the path f.

[35]

(c) If X, Y are two path connected topological spaces, can we conclude that the fundamental group of the product space $X \times Y$ is isomorphic to the product of the fundamental groups of X and Y? Prove your conclusion.

[40]

4. (a) Show that any continuous map $f : I \to S^1$ has a lift $\tilde{f} : I \to \mathbb{R}$. Also show that, given $x_0 \; \varepsilon \; \mathbb{R}$, with $e(x_0) = f(0)$, there is a unique $\tilde{f}$ with $\tilde{f}(0) = x_0$ where $e : \mathbb{R} \to S^1$ is defined by $e(t) = \exp(2\pi i t)$ for $t \; \varepsilon \; \mathbb{R}$. [40]

(b) Let f_0 and f_1 be equivalent paths in S^1 based at 1. If $\tilde{f}_0$ and $\tilde{f}_1$ are lifts with $\tilde{f}_0(0) = \tilde{f}_1(0)$, prove that $\tilde{f}_0(1) = \tilde{f}_1(1)$, [20]

(c) Under usual notations, show that $\Pi(S^1, 1) \cong \mathbb{Z}$. [40]

5. (a) Define a covering map.

Let X be a $G-$space. If the action of G on X is properly discontinuous, show that $p : X \to X|_G$ is a covering. [20]

(b) Let $p : \tilde{X} \to X$ be a covering map. Then prove:

i. p is an open map.

ii. X has the quotient topology with respect to p. [20]

(c) Let $p : \tilde{X} \to X$ be a covering with $\tilde{X}$ path connected. If $\tilde{x_0}, \tilde{x_1} \; \varepsilon \; \tilde{X}$, prove that there is a path f in X from $p(\tilde{x_0})$ to $p(\tilde{x_1})$ such that $U_f p_*(\Pi(\tilde{X}, \tilde{x_0})) = p_*(\Pi(\tilde{X}, \tilde{x_1}))$. [15]

(d) Establish the following results:

i. The function $\psi : \Pi(X|_G, y_0) \to G$ is a homomorphism of groups.

ii. The kernel of $\psi : \Pi(X|_G, \; y_0) \to G$ is a sub group $p_*(\Pi(X, x_0))$.

iii. $\Pi(X|_G, \; y_0)/_{p_*(\Pi(X, \; x_0))} \cong G$. [45]

6. (a) If X is a non-empty path connected space, prove that $H_0(X) \cong \mathbb{Z}$. [30]

(b) Under the usual notations, show that $\partial f_{\#} = f_{\#} \partial$. [20]

(c) Let $f, g : X \to Y$ be two continuous maps. If f and g are homotophic, show that $f_* = g_* : H_n(X) \to H_n(Y), \forall n > 0$. [50]