EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

SPECIAL DEGREE EXAMINATION

IN MATHEMATICS, (1998/1999)

(July/August 2005)

MT 418 - LIE GROUP ANALYSIS OF ORDINARY DIFFERENTIAL EQUATIONS

Answer all questions Time allowed: **3** Hours

Q1. (i) Show that, for every $\epsilon \in \mathbb{R}$,

$$(\,\overline{x}, \overline{y}\,) = \left(x,\; y + \epsilon \exp\left\{\int \mathrm{F(x)\,dx}\right\}\right)$$

is a symmetry of the general first-order linear ordinary differential equation (ODE)

$$\frac{dy}{dx} = F(x)y + G(x)\,.$$

[30 Marks]

(ii) Obtain the one parameter group of point transformations corresponding to the generator

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

[35 Marks]

(iii) Determine the most general scalar first-order ordinary differential equation (ODE) invariant under the group generated by the above symmetry generator. [35 Marks]

Q2. (i) The Riccati equation

$$y' = xy^2 - \frac{2y}{x} - \frac{1}{x^3}, \quad x \neq 0$$

has a Lie group of scaling symmetries

$$\overline{x} = e^{\epsilon} x \; ; \; \overline{y} = e^{-2\epsilon} y.$$

Use the above one parameter group of transformations to integrate the Riccati equation. [50 Marks]

(ii) Let the differential equation be

$$y'' + 3yy' + y^3 = 0.$$

Verify that the symmetry generator

$$X = xy\frac{\partial}{\partial x} + (y^2 - xy^3)\frac{\partial}{\partial y}$$

is one of the generators of the equation. Use the above generator to integrate the equation. [50 Marks]

Q3. (i) Show that any two-dimensional abelian Lie algebra $L_2 = \langle X_1, X_2 \rangle$ of generators is solvable. [40 Marks]

(ii) Obtain the Lie algebra of the generators

$$X_1 = \frac{\partial}{\partial x}, \;\; X_2 = \frac{\partial}{\partial y}, \;\; X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

[30 Marks]

(iii) Determine the first and second derived algebras of the algebra in (ii). [30 Marks]

Q4. The third-order ordinary differential equation

$$y''y''' - y''^2 - \exp 2x = 0$$

admits three generators of point symmetries

$$X_1 = \frac{\partial}{\partial y}, \;\; X_2 = x\frac{\partial}{\partial y}, \;\; X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

(i) Use the solvability of the Lie algebra to consecutively lower the order of the equation by three. [50 Marks]

(ii) Find the general solution of the third-order equation iterms of quadrature/s. [50 Marks]

Q5. The nonlinear second-order differential equation

$$y'' = y'(y'-1)^2 F(y-x),$$

where F is an arbitrary function of its argument, admits the symmetry generators

$$X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad X_2 = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Obtain a point transformation that reduces these generators to their canonical form. Use this transformation to reduce the original equation to a linear second-order equation. [100 Marks]

Q6. The second-order ordinary differential equation

$$y'' = x^{-5}y^2$$

admits the two generators of point symmetries

$$X_1 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

(i) Which of the generator/s above, if any, are Noether generator/s of point symmetry corresponding to the Lagrangian

$$L = \frac{1}{2}y'^2 + \frac{1}{3}x^{-5}y^3.$$

[50 Marks]

(ii) Use Noether's theorem to obtain the first integral/s. Hence use a Noether symmetry to reduce the second-order equation to a quadrature. [50 Marks]