EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

SPECIAL DEGREE EXAMINATION

IN MATHEMATICS, (1998/1999)

(July/August 2005)

MT 419 - LIE GROUP ANALYSIS OF PARTIAL DIFFERENTIAL EQUATIONS

Answer all questions | Time allowed: **3 Hours**

Q1. (i) Prolong the transformation G:

$$\overline{t} = t, \quad \overline{x} = x + 2at, \quad \overline{u} = ue^{-(ax+a^2t)}, \quad a \in \mathbb{R},$$

to the first derivatives. [30 Marks]

(ii) Prove that the first prolongation of the transformation G, denoted by $G^{[1]}$, forms a continuous one-parameter group of transformations. [40 Marks]

(iii) Deduce the generator $X^{[1]}$ corresponding to $G^{[1]}$. [30 Marks]

Q2. (i) Given the generator

$$X = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x},$$

obtain its one-parameter group by using the exponential map.

[30 Marks]

(ii) Show that the Sine-Gordon equation $u_{tt} - u_{xx} = \sin u$ has X in (i) as its generator of symmetry. [30 Marks]

(iii) Obtain the second-order ordinary differential equation which gives rise to the invariant solutions of the Sine-Gordon equation corresponding to X. [40 Marks]

Q3. The one-dimensional linear heat equation

$$u_t - u_{xx} = 0$$

admits amongst others the generator of point-symmetry

$$X = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} - \left[\frac{t}{2} + \frac{x^2}{4}\right] u\frac{\partial}{\partial u}.$$

Find the group-invariant solution corresponding to the symmetry generator X. [100 Marks]

Q4. (i) Show that the invariants of the operator $X = a\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}$, a is a constant, are $\gamma = x - \frac{t^2}{2a}$ and $\beta = u - \frac{t}{a}$. [40 Marks]

(ii) Given that the KdV (Korteweg-de Vries) equation

$$u_t + u_{xxx} + uu_x = 0$$

admits the symmetry generatot X above, show that the G-invariant solutions of the equation satify the ordinary differential equation

$$\beta_{\gamma\gamma} + \frac{1}{2}\beta^2 + \frac{1}{a}\gamma + c = 0,$$

c is a constant. [100 Marks]

Q5. The partial differential equation

$$u_t = (e^u u_x)_x$$

is a nonlinear heat equation admits the generators of point symmetries

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \quad X_4 = x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u}.$$

(i) Obtain the one-parameter groups G_i generated by the X_is. [40 Marks]

(ii) If $u = h(t, x)$ is a solution to the equation, find the solutions associated with each of the G_is. [30 Marks]

(iii) Find the most general travelling wave solutions to the equation. [30 Marks]

Q6. Equivalence transformation transforms any member of a class of partial differential equations to an equation which is also a member of the same class. Show that the transformation (ϕ and ψ are arbitrary functions)

$$\overline{x} = \phi(x), \quad \overline{y} = \psi(y), \quad \overline{u} = u,$$

is an equivalence tranformation of the hyperbolic equation

$$u_{xy} + A(x, y)u_x + B(x, y)u_y + C(x, y)u = 0,$$

where A, B and C are arbitrary functions. [100 Marks]