EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE 2002/2003

2002/2003(A) (Apr./May' 2004)

(Proper & Repeat)

SECOND SEMESTER

MT 202 - METRIC SPACE

Answer all questions

Time : Two hours

(a) Let X be the set of all bounded sequences of real numbers. Define $d : X \times X \longrightarrow \mathbb{R}$ by

$$d(x, y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$$

where $x = \{x_i\}_{i \in \mathbb{N}}$ and $y = \{y_i\}_{i \in \mathbb{N}}$ are two arbitrary elements of X. Show that (X, d) is a metric space. [20 marks]

(b) Prove that every open ball is an open set. [20 marks]

(c) Prove that, for any subset A of a metric space its interior A° is the largest open set contained in A. [25 marks]

(d) Is it true that, arbitrary union of closed sets is closed? Justify your answer. [10 marks]

(e) Prove that, if a subset F of a metric space is closed then F contains all of its limit points. [15 marks]

(f) Prove that, in any metric space singleton sets are closed sets. [10 marks]

2. (a) Let (Y, d) be a subspace of a metric space (X, d). Prove that $A \subseteq Y$ is open in Y if and only if there exists a set G open in X such that $A = Y \cap G$. [35 marks]

(b) In a metric space, prove that any Cauchy sequence that contains a convergent subsequence is convergent. [20 marks]

(c) Prove that the intersection of any collection of complete subsets of a metric space is complete. [20 marks]

(d) Is the open interval $(0, 1)$ compact in $\mathbb{R}$? Justify your answer. [25 marks]

3. (a) Prove that, two open sets are separated if and only if they are disjoint. [25 marks

(b) Prove that, a metric space M is disconnected if and only if there exists a no empty proper subset of M which is both open and closed. [25 mark

(c) Let (X, d) be a compact metric space. Prove that if A is a closed subset of then A is compact. [20 mar

(d) Prove that every compact subset of a metric space is bounded. Is the converse this result true? Justify your answer. [30 mar

4. What is meant by a function from a metric space (X, d) to a metric space (Y, ρ) is continuous at a point $a \in X$? [10 marks]

Let f be a function from a metric space (X, d) to a metric space (Y, ρ). Prove that the following statements are equivalent.

(i) f is continuous on X. [15marks]

(a) $f^{-1}(G)$ is an open subset of X whenever G is an open subset of Y. [20 marks]

(b) If $x_n \to x$ in X then $f(x_n) \to f(x)$ in Y. [20 marks]

(c) $f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A of X. [15 marks]

(d) $f^{-1}(C)$ is a closed subset of X whenever C is a closed subset of Y. [20 marks]