EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE 2002/2003 & 2002/2003(A) (Apr./May.' 2004)

SECOND SEMESTER

MT 217 - MATHEMATICAL MODELING

Answer all questions

Time : Two hours

1. Write down the steps involved in a mathematical model building process.

   Illustrate your mathematical model building process by considering the following example.

   It is about to rain; you have to walk a short distance of about 1Km between home and college. As there is some hurry, you do not bother to take a raincoat or umbrella. Suppose that it now starts to rain heavily and you do not turn back; how wet will you get?

2. (a) Explain the logistic model

$$\frac{dp}{dt} = ap - bp^2, \quad p(t_0) = p_0$$

of the population growth of a single species.

Prove that $\dfrac{a - bp_0}{a - bp(t)}$ is positive for $t_0 < t < \infty$ where $p(t_0) = p_0$.

Find $p(t)$ and the limiting value of $p(t)$, $t > t_0$.

(b) Assume that the worlds' resources will provide enough food only for 6×10^{10} humans, the worlds' population was 1.6×10^{10} in 1900 and 2.4×10^{10} in 1955. Using the logistic population model, predict the population for the year 2010.

3. The fish population in a certain part of the sea can be separated into prey population (food fish) $x(t)$ and predator population (Selachians) $y(t)$. The model governing the interaction of the selachians and food fish in the absence of fishing is given by

$$\left.\begin{aligned} \frac{dx}{dt} &= ax - bxy \\ \frac{dy}{dt} &= -cy + dxy \end{aligned}\right\} \cdots\cdots\cdots\cdots (1)$$

Explain the terms involved in this model.

Show that

i $\frac{y^a}{e^{by}} \cdot \frac{x^c}{e^{dx}} = k$, where k is a constant.

ii the solution of (1) defines a family of closed curves for $x, y > 0$.

Let $x(t)$ and $y(t)$ be the periodic solution of (1).
If $\overline{x} = \frac{1}{T}\int_0^T x(t)dt$ and $\overline{y} = \frac{1}{T}\int_0^T y(t)dt$ then show that $\overline{x} = \frac{c}{d}$ and $\overline{y} = \frac{a}{b}$ where T is the period.
Hence show that a moderate amount of fishing increases the average number of food fish and decreases the average number of selachians.

4. Explain the Pipe's model (P) given below (with the usual notations) for a line of traffic (car) in which all drivers obey California vehicle code:

"A good rule for following another vehicle at a safe distance is to allow yourself the length of a car (L^*ft) for every 10 miles per hour you are travelling".

$$\left.\begin{aligned} v_n(t) &= v_{n+1}(t) + T^* \frac{d}{dt} v_{n+1}(t) \\ T^* &= \frac{L^*}{14.67} \text{ Sec} \end{aligned}\right\} \cdots\cdots\cdots (\text{P})$$

Assuming $T^* = 1$, show that

$$v_{n+1}(t) = \frac{1}{(n-1)!} \int_0^t u^{n-1} e^{-u} v_1(t-u)\, du.$$

Suppose the lead vehicle is standing still at $t = 0$ and acquires a constant cruising speed v_c for $t > 0$. Show that

$$v_{n+1}(t) = v_c G_n(t)$$

where $G_n(t) = 1 - e^{-t} \left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \cdots + \frac{t^{n-1}}{(n-1)!}\right)$.

Show that there is no possibility of collision in this model.