

EASTERN UNIVERSITY, SRI LANKA
DEPARTMENT OF MATHEMATICS
SPECIAL DEGREE EXAMINATION IN MATHEMATICS
ACADEMIC YEAR - 2008/2009 (December, 2010)
Part II
MT 403 - ALGEBRAIC TOPOLOGY

Answer all questions.

Time allowed: Three hours

1. (a) Prove that a topological space (X, \mathcal{T}) is compact if and only if every collection of closed subsets of X with the finite intersection property has a non-empty intersection. [40 marks]
- (b) Prove that every compact topological space has the Bolzano-Weierstrass property, that is every infinite subset of the space has a limit point. [20 marks]
- (c) Prove the following:
 - i. Any continuous image of a compact space is compact. [20 marks]
 - ii. Any closed subspace of a compact space is compact. [20 marks]
2. (a) Prove that any compact subset of a Hausdorff space is closed. [25 marks]
- (b) Let f and g be continuous functions of a topological space (X, \mathcal{T}) into a Hausdorff space (Y, \mathcal{V}) . Prove that the set $\{x \in X : f(x) = g(x)\}$ is a closed subset of X . [30 marks]
- (c) Let (X, \mathcal{T}) be a Hausdorff space and let f be a continuous function of X into itself. Prove that the set of fixed points under f is a closed set. [30 marks]
- (d) Let K be a compact subset of a Hausdorff space X and suppose that p is a point in the complement of K . Show that there are disjoint open sets U and V with $p \in V$ and $K \subset U$. [15 marks]

3. (a) Let α be an equivalence class of paths with initial point x and terminal point y . Show that

$$\varepsilon_x \cdot \alpha = \alpha \quad \text{and} \quad \alpha \cdot \varepsilon_y = \alpha,$$

where ε_* is the equivalence class of the constant paths of I into $*$.

[30 marks]

- (b) If X is a path-connected space, then prove that the groups $\pi(X, x)$ and $\pi(X, y)$ are isomorphic for any pair of points $x, y \in X$.

[30 marks]

- (c) Prove that the image of a path-connected space under a continuous map is path-connected.

[20 marks]

- (d) Let $\{Y_i : i \in I\}$ be a collection of path-connected subsets of a space X . If $\bigcap_{i \in I} Y_i \neq \emptyset$, then show that $Y = \bigcup_{i \in I} Y_i$ is path-connected.

[20 marks]

4. (a) Prove that a space X is simply connected if and only if there is a unique homotopy class of paths connecting any two points in X .

[30 marks]

- (b) Prove that $\pi(X \times Y)$ is isomorphic to $\pi(X) \times \pi(Y)$ if X and Y are path-connected.

[40 marks]

- (c) If a space X retracts onto a subspace A , then show that the homomorphism $i_* : \pi(A, x_0) \rightarrow \pi(X, x_0)$ induced by the inclusion $i : A \rightarrow X$ is injective. If A is a deformation retract of X , then show that i_* is an isomorphism.

[10 marks]

- (d) Show that the map $\beta_h : \pi(X, x) \rightarrow \pi(X, x_0)$ defined by $\beta_h[f] = [h \cdot f \cdot \bar{h}]$ is an isomorphism.

[20 marks]

5. (a) If X is a simple point space, then prove that $H_0(X) \cong \mathbb{Z}$ and $H_n(X) = 0$ for $n > 0$.

[20 marks]

- (b) If X is a non-empty path-connected space, then prove that $H_0(X) \cong \mathbb{Z}$.

[40 marks]

- (c) Prove the following:

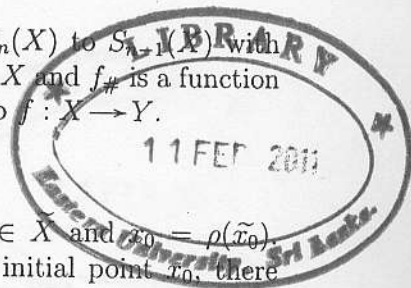
i. $\partial f_{\#} = f_{\#} \partial,$

[10 marks]

ii. $\partial \partial = 0,$

[30 marks]

where ∂ is the boundary operator from $S_n(X)$ to $S_{n-1}(X)$ with $S_n(X)$ being the set of singular n -chains in X and $f_\#$ is a function from $S_n(X)$ to $S_n(Y)$ for a continuous map $f : X \rightarrow Y$.



6. (a) Let (\tilde{X}, ρ) be a covering space of X , $\tilde{x}_0 \in \tilde{X}$ and $x_0 = \rho(\tilde{x}_0)$. Prove that for any path $f : I \rightarrow X$ with initial point x_0 , there exists a unique path $g : I \rightarrow \tilde{X}$ with initial point \tilde{x}_0 such that $\rho \circ g = f$. [20 marks]
- (b) Let (\tilde{X}, ρ) be a covering space of X and let $g_0, g_1 : I \rightarrow \tilde{X}$ be paths in \tilde{X} which have the same initial point. If $\rho \circ g_0 \sim \rho \circ g_1$, then show that $g_0 \sim g_1$. Also show that g_0 and g_1 have the same terminal point. [40 marks]
- (c) Let (\tilde{X}_1, ρ_1) and (\tilde{X}_2, ρ_2) be covering spaces of X and let ϕ be a homomorphism of the first covering space into the second. Then show that (\tilde{X}_1, ϕ) is a covering space of \tilde{X}_2 . [40 marks]