EASTERN UNIVERSITY, SRI LANKA
DEPARTMENT OF MATHEMATICS
SPECIAL DEGREE EXAMINATION IN MATHEMATICS
ACADEMIC YEAR - 2008/2009 (December, 2010)
Part II
MT 403 - ALGEBRAIC TOPOLOGY

Answer all questions. Time allowed: Three hours

---

1. (a) Prove that a topological space $(X, \mathcal{T})$ is compact if and only if every collection of closed subsets of X with the finite intersection property has a non-empty intersection. [40 marks]

   (b) Prove that every compact topological space has the Bolzano-Weierstrass property, that is every infinite subset of the space has a limit point. [20 marks]

   (c) Prove the following:

      i. Any continuous image of a compact space is compact. [20 marks]

      ii. Any closed subspace of a compact space is compact. [20 marks]

2. (a) Prove that any compact subset of a Hausdorff space is closed. [25 marks]

   (b) Let f and g be continuous functions of a topological space $(X, \mathcal{T})$ into a Hausdorff space $(Y, \mathcal{V})$. Prove that the set $\{x \in X : f(x) = g(x)\}$ is a closed subset of X. [30 marks]

   (c) Let $(X, \mathcal{T})$ be a Hausdorff space and let f be a continuous function of X into itself. Prove that the set of fixed points under f is a closed set. [30 marks]

   (d) Let K be a compact subset of a Hausdorff space X and suppose that p is a point in the complement of K. Show that there are disjoint open sets U and V with $p \in V$ and $K \subset U$. [15 marks]

3. (a) Let α be an equivalence class of paths with initial point x and terminal point y. Show that

$$\varepsilon_x.\alpha = \alpha \quad \text{and} \quad \alpha.\varepsilon_y = \alpha,$$

where ε_* is the equivalence class of the constant paths of I into $*$.

[30 marks]

(b) If X is a path-connected space, then prove that the groups $\pi(X, x)$ and $\pi(X, y)$ are isomorphic for any pair of points $x, y \in X$.

[30 marks]

(c) Prove that the image of a path-connected space under a continuous map is path-connected. [20 marks]

(d) Let $\{Y_i : i \in I\}$ be a collection of path-connected subsets of a space X. If $\bigcap_{i\in I} Y_i \neq \phi$, then show that $Y = \bigcup_{i\in I} Y_i$ is path-connected.

[20 marks]

4. (a) Prove that a space X is simply connected if and only if there is a unique homotopy class of paths connecting any two points in X.

[30 marks]

(b) Prove that $\pi(X \times Y)$ is isomorphic to $\pi(X) \times \pi(Y)$ if X and Y are path-connected. [40 marks]

(c) If a space X retracts onto a subspace A, then show that the homomorphism $i_* : \pi(A, x_0) \longrightarrow \pi(X, x_0)$ induced by the inclusion $i : A \longrightarrow X$ is injective. If A is a deformation retract of X, then show that i_* is an isomorphism. [10 marks]

(d) Show that the map $\beta_h : \pi(X, x) \longrightarrow \pi(X, x_0)$ defined by $\beta_h[f] = [h.f.\bar{h}]$ is an isomorphism. [20 marks]

5. (a) If X is a simple point space, then prove that $H_0(X) \cong \mathbb{Z}$ and $H_n(X) = 0$ for $n > 0$. [20 marks]

(b) If X is a non-empty path-connected space, then prove that $H_0(X) \cong \mathbb{Z}$. [40 marks]

(c) Prove the following:

i. $\partial f_\# = f_\# \partial$, [10 marks]

ii. $\partial\partial = 0$, [30 marks]

where ∂ is the boundary operator from $S_n(X)$ to $S_{n-1}(X)$ with $S_n(X)$ being the set of singular n-chains in X and $f_\#$ is a function from $S_n(X)$ to $S_n(Y)$ for a continuous map $f : X \longrightarrow Y$.

6. (a) Let $(\widetilde{X}, \rho)$ be a covering space of X, $\widetilde{x_0} \in \widetilde{X}$ and $x_0 = \rho(\widetilde{x_0})$. Prove that for any path $f : I \longrightarrow X$ with initial point x_0, there exists a unique path $g : I \longrightarrow \widetilde{X}$ with initial point $\widetilde{x_0}$ such that $\rho.g = f$. [20 marks]

   (b) Let $(\widetilde{X}, \rho)$ be a covering space of X and let $g_0, g_1 : I \longrightarrow \widetilde{X}$ be paths in $\widetilde{X}$ which have the same initial point. If $\rho.g_0 \sim \rho.g_1$, then show that $g_0 \sim g_1$. Also show that g_0 and g_1 have the same terminal point. [40 marks]

   (c) Let $(\widetilde{X_1}, \rho_1)$ and $(\widetilde{X_2}, \rho_2)$ be covering spaces of X and let ϕ be a homomorphism of the first covering space into the second. Then show that $(\widetilde{X_1}, \phi)$ is a covering space of $\widetilde{X_2}$. [40 marks]