EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

SPECIAL DEGREE EXAMINATION IN MATHEMATICS

ACADEMIC YEAR - 2008/2009 (December, 2010)

Part II

MT 408 - RELATIVITY

Answer all questions Time allowed: Three hours

1. In two spacetime dimensions two observers moving with constant relative velocity v set up coordinate system (ct, x) and (ct', x') respectively. Show that if they set their clocks to $t = t' = 0$ when pass each other, the transform between these coordinate systems is the Lorentz transform:

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \gamma \begin{pmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}, \text{ where } \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}.$$

Let K and K' be two inertial frames such that the origin of K' moves with relative speed v in the $x-$direction.

   (a) In K a rod at rest has length l_0. What is the length of the rod in K'?

   (b) Let A and B be two simultaneous events in K and suppose A is at $(0, 0)$ and B is at $(0, x)$ where $x \neq 0$. Show that A and B are not simultaneous in K'.

   (c) Show that a particle moving with the speed of light along the $x-$direction in K also moves at the speed of light in K'.

2. (a) A body with rest mass m disintegrates at rest into two parts with rest mass m_1 and m_2. Show that the energies of the two parts are

$$E_1 = c^2 \frac{m^2 + m_1^2 - m_2^2}{2m}, \quad E_2 = c^2 \frac{m^2 + m_2^2 - m_1^2}{2m}.$$

(b) A photon with energy $E = h\nu$ collides with an electron of mass m which is initially at rest. After the collision the photon is scattered by an angle θ with energy $E' = h\nu'$, while the electron moves off at an angle ϕ to the photon's original path.

i. Show that

$$\lambda' - \lambda = \frac{2h}{mc}\sin^2\frac{\theta}{2},$$

where λ and λ' are the wavelength of the photon before and after collision.

ii. Show that

$$\tan\phi = \frac{\cot\frac{\theta}{2}}{1 + \frac{h\nu}{mc^2}}$$

3. (a) Show that the Riemann tensor

$$R^d{}_{abc} = \Gamma^d{}_{ac,b} - \Gamma^d{}_{ab,c} + \Gamma^e{}_{ac}\Gamma^d{}_{eb} - \Gamma^e{}_{ab}\Gamma^d{}_{ec}$$

arises from the equation

$$V_{a;bc} - V_{a;cb} = R^d{}_{abc}V_d$$

(b) Using the Bianchi identity

$$R^a{}_{bcd;e} + R^a{}_{bde;c} + R^a{}_{bec;d} = 0$$

show that $G^{ab}{}_{;b} = 0$.

(c) Prove the following:

i. $R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc} = 0$;

ii. $k_{a;bc} = R_{abcd}k^d$ if $k_{a;b} + k_{b;a} = 0$.

4. (a) Show that in a 2-dimensional Riemannian manifold all components of R_{abcd} are either zero or $\pm R_{1212}$.

(b) In terms of the usual polar angles, the metric tensor field of a sphere of radius a is given by

$$[g_{ab}] = \begin{bmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{bmatrix}.$$

Show that $R_{1212} = a^2\sin^2\theta$ and deduce that

$$[R_{ab}] = \begin{bmatrix} -1 & 0 \\ 0 & -\sin^2\theta \end{bmatrix}$$

also establish that $R = -2/a^2$.

5. Use the Euler-Lagrange equations to obtain the geodesic equations and hence show that the only non-vanishing Ricci tensors are given by

$$\begin{aligned} R_{00} &= \frac{A''}{2B} - \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{rB} \\ R_{11} &= -\frac{A''}{2A} + \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{B'}{rB} \\ R_{22} &= -\frac{1}{B} + 1 - \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) \\ R_{33} &= R_{22}\sin^2\theta \end{aligned}$$

for the line element of a spherically symmetric spacetime

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

6. (a) Using the result obtained in question 5, generate the exterior Schwarzschild solution

$$ds^2 = -c^2\left(1 - \frac{2GM}{c^2 r}\right)dt^2 + \left(1 + \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

clearly stating all results used.

(b) Given the equations for a particle

$$\begin{aligned} \left(1 - \frac{2m}{r}\right)\dot{t} &= k \\ r^2\dot{\phi} &= h \\ c^2\left(1 - \frac{2m}{r}\right)\dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 &= c^2 \end{aligned}$$

show that the following results hold in vertical free fall:

$$\begin{aligned} k &= \sqrt{1 - 2m/r_0} \\ \ddot{r} + GM\frac{1}{r^2} &= 0 \\ \frac{1}{2}\dot{r}^2 &= MG\left(\frac{1}{r} - \frac{1}{r_0}\right) \end{aligned}$$

where r_0 is the point of release of the particle.