

EASTERN UNIVERSITY, SRI LANKA

THIRD YEAR EXAMINATION IN SCIENCE 2002/2003

SECOND SEMESTER

(April/May '2004)

(Re-Repeat)

MT 302 - COMPLEX ANALYSIS

---

Answer five questions only

Time : Three hours

---

1. (a) Define what is meant by “ a complex-valued function  $f$  has a limit at  $z_0 \in \mathbb{C}$  ”.

(b) Show that

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \quad \text{for } |z| < 1.$$

Deduce that if  $0 < r < 1$ , then

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r \cos \theta - r^2}{1 - 2r \cos \theta + r^2},$$

$$\sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r \sin \theta - r^2}{1 - 2r \cos \theta + r^2}.$$

2. (a) What is meant by saying that a complex-valued function  $f$ , defined on a domain  $D (\subseteq \mathbb{C})$ , is analytic at a point  $z_0 \in D$ .

Show that if  $z = x + iy$  and a function  $f(z) = u(x, y) + iv(x, y)$  is

analytic at  $z_0 = x_0 + iy_0$ , then the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are satisfied at every point of some neighbourhood of  $z_0$ .

- (b) Prove that the function  $u(x, y) = 3x^2y + 2x^2 - y^3 - 2y^2$  is harmonic.

Find a function  $v(x, y)$  such that  $f(z) = u(x, y) + iv(x, y)$  is analytic.

3. Let  $M > 0$  be such that  $|f(z)| \leq M$  for all  $z$  on a contour  $C$  and  $l$  be the length of  $C$ .

Show that

$$\left| \int_C f(z) dz \right| \leq Ml.$$

Hence show that

$$\left| \int_C \frac{z^{1/2}}{z^2 + 1} dz \right| \leq \frac{3\sqrt{3}}{8}\pi,$$

where  $C$  is the semi circular path given by  $z = 3e^{i\theta}$ ,  $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ .

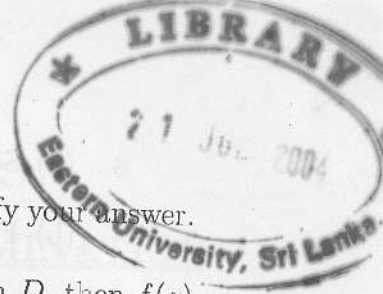
4. Let  $f$  be analytic everywhere within and on a simple closed contour  $C$ , taken in the positive sense. If  $z_0$  is any point interior to  $C$ , then prove that

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz.$$

Prove that if  $f(z)$  is analytic inside and on the circle  $C$  of radius  $r$  with centre at  $z = z_0$ , then

$$\left| f^{(n)}(z_0) \right| \leq \frac{Mn!}{r^n}, \quad n = 0, 1, 2, \dots$$

where  $M$  is a positive constant such that  $|f(z)| \leq M$  for all  $z$  inside and on  $C$ .



5. Prove or disprove each of the following statements. Justify your answer.

(a) If  $f(z)$  and  $\overline{f(z)}$  are analytic functions in a domain  $D$ , then  $f(z)$  is a constant in  $D$ .

(b) The function  $f(z) = \frac{1}{z}$  is uniformly continuous in  $|z| < 1$ .

(c) Every polynomial of degree  $n$  with complex coefficients, has exactly  $n$  zero.

(d) The function  $f : \mathbb{C} \rightarrow \mathbb{C}$  defined by  $f(z) = |z|^2$ , has derivative at each point in  $\mathbb{C}$ .

6. (a) Let  $f$  be a complex-valued function and  $z_0 \in \mathbb{C}$ . Explain what is meant by each of the following statements:

(i)  $f$  has a pole of order  $m$  at  $z_0$ ;

(ii) residue of  $f$  at  $z_0$ .


(b) Show that if  $f$  is analytic inside and on a simple closed contour  $C$  and  $f$  has a pole of order  $m$  at  $z = \alpha$  then the residue of  $f$  at  $z = \alpha$  is given by

$$\lim_{z \rightarrow \alpha} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z - \alpha)^m f(z) \right\}.$$

7. (a) State and prove the Argument Theorem.

(b) (i) If  $f(z)$  and  $g(z)$  are analytic inside and on a simple closed curve  $C$  and if  $|g(z)| < |f(z)|$  on  $C$ , then show that both functions  $f(z) + g(z)$  and  $f(z)$  have the same number of zeros inside  $C$ .

(ii) Show that all the roots of  $2z^5 - z^3 + z + 7 = 0$  lie between the circles  $|z| = 1$  and  $|z| = 2$ .



8. (a) State the Residue Theorem.

(b) Find the value of the integral

$$\oint_C \frac{3z^2 + 2}{(z-1)(z^2+9)} dz$$

where  $C$  is taken counter clockwise around the circle

(i)  $|z-2| = 2$  ;

(ii)  $|z| = 4$ .