EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE (2002/2003)

SECOND SEMESTER

MT 307 - CLASSICAL MECHANICS III

Answer all questions Time: Three hours

1. Obtain the equation of motion of a particle of mass m which moves on a smooth horizontal plane of the form

$$\frac{\partial^2 \underline{r}}{\partial t^2} + 2\underline{\omega} \wedge \frac{\partial \underline{r}}{\partial t} = \left(\frac{R}{m} - g\right) \underline{Z}$$

fixed on the earth's surface, where ω is the angular velocity of the earth, R is the normal reaction on the particle, $\underline{r}$ is the position vector of the particle, $\underline{Z}$ is the unit normal vector to the plane.
Derive the equation,

$$\frac{\partial^2 \underline{r}}{\partial t^2} + 2\omega \sin \lambda \underline{Z} \wedge \frac{\partial \underline{r}}{\partial t} = 0$$

and hence show that if the particle is projected with velocity

$$v_0 = 2\omega \sin \lambda (\underline{Z} \wedge \underline{a}),$$

where $\underline{a}$ is the position vector of a fixed point on the plane, the path of particle is a circle with centre at $\underline{a}$, where λ is the latitude.
Further if $\underline{v}_0 = u_0 \underline{i} + \omega_0 \underline{j}$, then show that the normal reaction R on particle is given by

$$R = mg - 2m\omega \cos \lambda u_0,$$

where $\underline{i}$, $\underline{j}$ are unit vectors along the east and north respectively.

2. (a) With usual notation, obtain the equations

i. $\dfrac{d\underline{H}}{dt} = \displaystyle\sum_{i=1}^{N} \underline{r}_i \wedge \underline{F}_i;$

ii. $\dfrac{d\underline{H_G}}{dt} = \displaystyle\sum_{i=1}^{N} \underline{R_i} \wedge \underline{F_i}$

for a system of N particles moving in space.

(b) Consider a solid cylinder of mass m and radius a slipping without rolling down a smooth inclined face of a wedge of mass M that is free to move on a smooth horizontal plane.

i. How far has the wedge moved by the time that the cylinder has descended a vertical distance h from rest?

ii. Now suppose that the cylinder is free to roll down the wedge without slipping.

How far does the wedge move in this case?

iii. In which case does the cylinder reach the bottom faster? How does this depend on the radius of the cylinder?

3. (a) With usual notation, obtain,

$$s + \dot{\phi}\cos\theta = \text{ constant } = n,$$
$$A\dot{\phi}\sin^2\theta + Cn\cos\theta = \text{ constant } = k,$$
$$A(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + 2mgh\cos\theta = \text{ constant},$$

for the motion of a top with its tip on a perfectly rough horizontal floor, where s is the spin angular velocity of the top.

(b) Let $u = \cos\theta$. Prove that

i. $\dot{u}^2 = (\alpha - \beta u)(1 - u^2) - (\gamma - \delta u)^2 = f(u)$ (say)

where $\alpha = \dfrac{2E - Cn^2}{A}$, $\beta = \dfrac{2mgh}{A}$, $\gamma = \dfrac{k}{A}$ and $\delta = \dfrac{Cn}{A}$.

ii. $t = \displaystyle\int \frac{du}{\sqrt{f(u)}} + \text{ constant}.$

4. With usual notations deduce Lagrange's equations for impulsive motion from Lagrange's equations for a holonomic system in the form

$$\left(\frac{\partial T}{\partial \dot{q}_j}\right)_2 - \left(\frac{\partial T}{\partial \dot{q}_j}\right)_1 = S_j, \qquad j = 1, 2, \cdots,$$

where subscripts 1 and 2 denote quantities before and after the application of the impulse respectively.

Two rods AB and BC each of length a and mass m, are smoothly joined at B and the system lie on a frictionless horizontal table. Initially the points A, B and C are colinear. An impulse I is applied at A in a direction perpendicular to the line ABC.

(a) Find the equations of motion.

(b) Prove that, immediately after the application of impulse,

i. the centre of mass of BC has velocity of magnitude $\frac{I}{4m}$,

ii. the centre of mass of AB has velocity of magnitude $\frac{5I}{4m}$.

5. Define Hamiltonian functions in terms of Lagrangian function. Show, with the usual notations, that the Hamiltonian equations are given by,

$$(a)\ \dot{q}_j = \frac{\partial H}{\partial P_j}, \qquad (b)\ \dot{p}_j = -\frac{\partial H}{\partial q_j} \qquad (c)\ \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

and that, for a function $G(p, q, t)$,

$$\frac{dG}{dt} = [G, H] + \frac{\partial G}{\partial t}.$$

Prove the Poisson's theorem that $[f, g]$ is a constant of motion when f and g are constants of motion.

For a certain system with two degrees of freedom, the Hamiltonian is given by

$$H = \eta^2(p_1^2 + p_2^2) + \nu^2(p_1 q_1 + p_2 q_2)^2$$

where η and ν are constants.

Show that H is a constant of motion and that if $F = p_1 q_1 + p_2 q_2$, then

$$[F, H] = 2(H - \nu^2 F^2)$$

Show also that

$$F = \frac{\sqrt{H}}{\nu} \tanh 2\nu\sqrt{H}(t - t_0),$$

where t_0 is a constant.

6. Explain what is meant by

(a) the normal mode,

(b) the normal co-ordinates

of a dynamical system.

A uniform bar of length l and mass m is suspended from its ends by identical springs of elastic constant k. Motion is initiated by depressing one end by a small distance a and releasing from rest. Solve this problem and find the normal modes.