EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST EXAMINATION IN SCIENCE -(2007/2008)

SECOND SEMESTER (Aug/Sept, 2009)

MT 102 - REAL ANALYSIS

(PROPER/REPEAT)

Answer all questions — Time: Three hours

Q1. (a) Define the terms Supremum and Infimum of a bounded susset of $\mathbb{R}$.

Find the Supremum and Infimum of each subset of $\mathbb{R}$. State whether they are in S.

i. $S = \{\frac{1}{n} : n \in \mathbb{N}\}$;

ii. $S = \{x \in \mathbb{R} : |2x+1| < 5\}$;

iii. $S = \{x \in \mathbb{Q} : x^2 \leq 7\}$.

[40 Marks]

(b) State and prove the Archimedian property. Prove that between any two distinct real numbers, there exists a rational number and an irrational number. [40 Marks]

(c) State the mathematical induction principle and use it to prove that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

for all positive integers n. [20 Marks]

Q2. (a) State the following theorems with reference to a sequence of real numbers.

i. Monotone convergence theorem;

ii. Monotone subsequence theorem;

iii. Bolzano-Weierstras theorem.

Prove the Bolzano-Weierstras theorem from the Monotone convergence theorem and Monotone subsequence theorem. [50 Marks]

(b) Let (y_n) be a sequence of real numbers defined by $y_1 = 1$, and $y_n = \frac{1}{4}(2y_n + 3), \ \forall n \in \mathbb{N}$. Show that this sequence is convergent. Also find the limit of this sequence. [50 Marks]

Q3. (a) Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function. Define what it means to say that the limit of f at a point x_0 is l (i.e, $\lim_{x \to x_0} f(x) = l$).
By varifying the appropriate definitions prove

i. $\lim_{x \to 2} (2x^2 - x + 1) = 7$;

ii. $\lim_{x \to \infty} \dfrac{1}{x^2 + 2x + 1} = 0.$

[30 Marks]

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Assume that $\lim_{x \to a} f(x) = l$ and $l \neq 0$.Prove that there exists $\delta > 0$ such that $\dfrac{|l|}{2} < |f(x)| < \dfrac{3|l|}{2}$ for all x satisfying $0 < |x - a| < \delta$.
Prove that if $\lim_{x \to a} f(x) = l$ with $l \neq 0$ then $\lim_{x \to a} \dfrac{1}{f(x)} = \dfrac{1}{l}$. [40 Marks]

(c) If $\lim_{x \to a} f(x) = l$, then show that $\lim_{x \to a} |f(x)| = |l|$. Also give an example to show that the converse part of the above result is not true. [30 Marks]

Q4. (a) Let $f : A \to \mathbb{R}$ and $a \in \mathbb{R}$, where $A \subseteq \mathbb{R}$. When we say that f is differentiable at 'a'.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin\ x \quad \forall x \in \mathbb{R}$. Use the definition to prove that f is differentiable at every point $a \in \mathbb{R}$, and $f'(a) = \cos\ a$.

[30 Marks]

(b) Let $f, g : A \to \mathbb{R}$ both be differentiable at $a \in \mathbb{A}$, where $A \subseteq \mathbb{R}$. Prove, using the rules of limit that $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$. [30 Marks]

(c) State the Mean value theorem.

Use the mean value theorem to prove the following:

i. $\sin\ x < x$ for $x > 0$;

ii. $\ln(1 + x) < x$ for $x > 0$;

Deduce that $e^{-x} \sin\ x < \frac{x}{1+x}$ for $x > 0$. [40 Marks]

Q5. (a) Explain in terms of ε, δ what it means to say that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point x_0.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \cos\ x \quad \forall x \in \mathbb{R}$. Prove that f is continuous at every point $a \in \mathbb{R}$. [30 Marks]

(b) State Rolle's theorem.

Let $f, g : [a, b] \to \mathbb{R}$ be two functions with $a < b$. Suppose that f and g are differentiable on (a, b) and continuous on $[a, b]$ and that $g'(x) \neq 0$ for all $x \in (a, b)$.

Prove that there exists $c \in (a, b)$ for which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

[30 Marks]

(c) Calculate the following limits stating carefully any results you use.

i. $\lim_{\theta \to 0} \dfrac{\theta + \tan\,\theta}{\sin\,\theta}$;

ii. $\lim_{x \to 0} \dfrac{\ln^2(1+x) + \ln^2(1-x)}{x^2}$.

[40 Marks]

Q6. (a) Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of real numbers. What do you mean by the following:

i. $\lim_{n \to \infty} (a_n) = L$, where L is a real number;

ii. $\lim_{n \to \infty} (a_n) = \infty$;

iii. (a_n) is a Cauchy sequence.

[30 Marks]

(b) Use the definition to show that $\lim_{n \to \infty} \left(\dfrac{3n+2}{n+1} \right) = 3$. [30 Marks]

(c) Prove that every cauchy sequence of real numbers is bounded.

[20 Marks]

(d) Prove that the sequence (x_n) given by $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$; $\forall\, n \in \mathbb{N}$ is not cauchy.

[20 Marks]