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Eastern University, Sri Lanka

EASTERN UNIVERSITY, SRI LANKA  
DEPARTMENT OF MATHEMATICS  
FIRST EXAMINATION IN SCIENCE -2007/2008

SECOND SEMESTER (Aug/Sept., 2009)

MT 105 - THEORY OF SERIES

(PROPER)

Answer all Questions

Time: Two hours

1. (a) Define what is meant by the convergent or divergent of an infinite series  $\sum_{n=1}^{\infty} a_n$ .

Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} + \dots,$$

is convergent and find its sum.

- (b) Let  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  be two series of real numbers.

i. Show that if  $\sum_{n=1}^{\infty} a_n$  converges, then  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ .

ii. Is it true that, if  $a_n \rightarrow 0$  as  $n \rightarrow \infty$  then the series  $\sum_{n=1}^{\infty} a_n$  converges?

Justify your answer.

2. (a) Let  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  be series of positive real numbers such that  $\left(\frac{a_n}{b_n}\right)$  tends to a finite non-zero limit as  $n \rightarrow \infty$ . Prove that  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  either both converge or both diverge.

(b) Determine whether the following series converge or diverge:

i.  $2 + \frac{3}{2^3} + \frac{4}{3^3} + \frac{5}{4^3} + \dots$ ,

ii.  $1 + \frac{2^2 + 1}{2^3 + 1} + \frac{3^2 + 1}{3^3 + 1} + \frac{4^2 + 1}{4^3 + 1} + \dots$ .

(c) i. Let  $(a_n)_{n=1}^{\infty}$  be a decreasing sequence of positive terms such that  $a_n \rightarrow 0$  as  $n \rightarrow \infty$ . Show that the series  $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$  converges.

ii. Prove that  $\sum_{n=1}^{\infty} (-1)^{n+1} \sin\left(\frac{1}{n}\right)$  converges. What will happen to this series if we drop the factor  $(-1)^{n+1}$ ? Justify your answer.

3. (a) Define the following terms:

i. absolutely convergent;

ii. conditionally convergent.

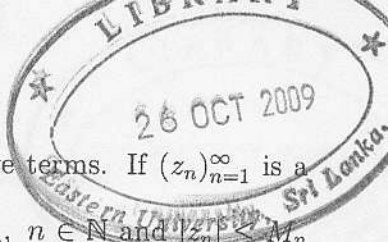
(b) i. Let  $\sum_{n=1}^{\infty} a_n$  be a series of real numbers. Prove that, if  $\sum_{n=1}^{\infty} |a_n|$  converges then  $\sum_{n=1}^{\infty} a_n$  also converges.

ii. Is it true that the rearrangement of a conditionally convergent series can change its sum? Justify your answer.

(c) i. If a power series  $\sum_{n=0}^{\infty} c_n x^n$  converges for  $x = x_0$  then show that it is absolutely convergent for every  $x = x_1$ , where  $|x_1| < |x_0|$ .

ii. Find the interval of convergence for the following power series

$$\sum_{n=1}^{\infty} \frac{(3x - 2)^n}{5^n}.$$



4. (a) Let  $\sum_{n=1}^{\infty} M_n$  be a convergent series of real non-negative terms. If  $(z_n)_{n=1}^{\infty}$  is a sequence of complex numbers such that  $z_n = x_n + iy_n$ ,  $n \in \mathbb{N}$  and  $|z_n| \leq M_n$

for all  $n \in \mathbb{N}$ , then show that  $\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} (x_n + iy_n)$  converges.

Hence check whether the series  $\sum_{n=1}^{\infty} \frac{(n+i)(1+ni)}{n^2}$  converges or diverges.

(b) If a series  $\sum_{n=1}^{\infty} z_n$  is such that  $\lim_{n \rightarrow \infty} \sqrt[n]{|z_n|} = l$ , then prove that

i. if  $l < 1$  the series converges absolutely,

ii. if  $l > 1$  the series diverges.

Hence check whether the series  $\sum_{n=0}^{\infty} \left(\frac{1}{2+i}\right)^n$  converges or diverges.