EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST EXAMINATION IN SCIENCE -2007/2008

SECOND SEMESTER (Aug/Sept., 2009)

MT 105 - THEORY OF SERIES

(PROPER)

Answer all Questions Time: Two hours

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1. (a) Define what is meant by the convergent or divergent of an infinite series $\sum_{n=1}^{\infty} a_n$.

   Show that the series

   $$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \frac{1}{7\cdot 9} + \cdots,$$

   is convergent and find its sum.

   (b) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of real numbers.

   i. Show that if $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$ as $n \to \infty$.

   ii. Is it true that, if $a_n \to 0$ as $n \to \infty$ then the series $\sum_{n=1}^{\infty} a_n$ converges? Justify your answer.

2. (a) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series of positive real numbers such that $\left(\frac{a_n}{b_n}\right)$ tends to a finite non-zero limit as $n \to \infty$. Prove that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

(b) Determine whether the following series converge or diverge:

i. $2 + \frac{3}{2^3} + \frac{4}{3^3} + \frac{5}{4^3} + \cdots$,

ii. $1 + \frac{2^2+1}{2^3+1} + \frac{3^2+1}{3^3+1} + \frac{4^2+1}{4^3+1} + \cdots$.

(c) i. Let $(a_n)_{n=1}^{\infty}$ be a decreasing sequence of positive terms such that $a_n \to$ as $n \to \infty$. Show that the series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

ii. Prove that $\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(\frac{1}{n}\right)$ converges. What will happen to this seri if we drop the factor $(-1)^{n+1}$? Justify your answer.

3. (a) Define the following terms:

i. absolutely convergent;

ii. conditionally convergent.

(b) i. Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers. Prove that, if $\sum_{n=1}^{\infty} |a_n|$ converg then $\sum_{n=1}^{\infty} a_n$ also converges.

ii. Is it true that the rearrangement of a conditionally convergent series ca change its sum? Justify your answer.

(c) i. If a power series $\sum_{n=0}^{\infty} c_n x^n$ converges for $x = x_0$ then show that it is abs lutely convergent for every $x = x_1$, where $|x_1| < |x_0|$.

ii. Find the interval of convergence for the following power series

$$\sum_{n=1}^{\infty} \frac{(3x-2)^n}{5^n}.$$

4. (a) Let $\sum_{n=1}^{\infty} M_n$ be a convergent series of real non-negative terms. If $(z_n)_{n=1}^{\infty}$ is a sequence of complex numbers such that $z_n = x_n + iy_n,\ n \in \mathbb{N}$ and $|z_n| \leq M_n$ for all $n \in \mathbb{N}$, then show that $\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty}(x_n + iy_n)$ converges.

Hence check whether the series $\sum_{n=1}^{\infty} \frac{(n+i)(1+ni)}{n^2}$ converges or diverges.

(b) If a series $\sum_{n=1}^{\infty} z_n$ is such that $\lim_{n\to\infty} \sqrt[n]{|z_n|} = l$, then prove that

i. if $l < 1$ the series converges absolutely,

ii. if $l > 1$ the series diverges.

Hence check whether the series $\sum_{n=0}^{\infty} \left(\frac{1}{2+i}\right)^n$ converges or diverges.