EASTERN UNIVERSITY, SRI LANKA

SPECIAL DEGREE EXAMINATION IN MATHEMATICS

2001/2002 (Jan/Feb.'2004)

MT 407 - RING THEORY

You should answer all questions. Time allowed is THREE hours only. Each question carries ONE HUNDRED marks.

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1. (a) Prove that every finite integral domain is a field. [25 marks]

   (b) If R is a commutative ring with unity, show that an ideal M in R is maximal if and only if R/M is a field. [40 marks]

   (c) Let R be a principal ideal domain. Prove that every non-zero non-unit element of R can be expressed as a product of irreducible elements.

   [35 marks]

2. (a) If F is a field, prove that $F[x]$ is a Euclidean domain.

   [40 marks]

   (b) State and prove Eisenstein's criterion. [25 marks]

   (c) Let R be a unique factorization domain [ufd]. Show that the polynomial ring $R[x]$ is also a ufd. [35 marks]

3. (a) Prove that an ideal $< p >$ in a principal ideal domain is maximal if and only if p is irreducible. [40 marks]

(b) Show that the set $\mathbb{Z}[i]$ of Gaussian integers is a Euclidean domain. [30 marks]

(c) Show, by an example, that every integral domain need not be a unique factorization domain. [30 marks]

4. (a) In a ring $\mathbb{Z}$ of integers, let p be a prime integer and let that, for some integer c, relatively prime to p, there exist integers x and y such that $x^2 + y^2 = cp$. Prove that integers a and b be found such that $p = a^2 + b^2$. [40 marks]

(b) Let M be an $R-$module and $x \in M$. Show that the set $K = \{ rx + nx \mid r \in R, n \in \mathbb{Z} \}$ is an $R-$submodule of M containing x. [30 marks]

(c) Prove that the submodules of the quotient module M/N are of the form U/N where U is a submodule of M containing N. [30 marks]

5. (a) Let R be a ring with unity and $\text{Hom}_R(R, R)$ denote the ring of endomorphisms of R regarded as a right $R-$module. Prove that $R \simeq \text{Hom}_R(R, R)$ as rings. [30 marks]

(b) Let M be a finitely generated free module over a commutative ring R. Show that all bases of M have the same number of generators. [30 marks]

(c) Prove, for an $R-$module M, that the following are equivalent:

i. M is noetherian;

ii. Every submodule of M is finitely generated;

iii. Every nonempty set S of submodules of M has a maximal element.

[40 marks]

6. (a) Let R be a principal ideal domain and M be a free module of rank n over R and N be a submodule of M. Prove that N is a free module of rank $\leq n$.

[45 marks]

(b) Let M be a finitely generated module over a principal ideal domain. Prove that M can be expressed as $M = F \oplus t(M)$ where F is a free submodule of M and $t(M)$ is the torsion submodule of M, and F is unique up to isomorphism.

[55 marks]