

EASTERN UNIVERSITY, SRI LANKA

SPECIAL DEGREE EXAMINATION IN MATHEMATICS

2001/2002 (Jan/Feb.'2004)

MT 407 - RING THEORY

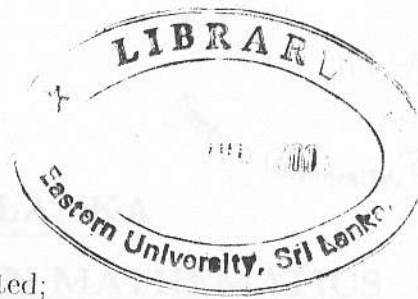
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You should answer all questions. Time allowed is THREE hours only.  
Each question carries ONE HUNDRED marks.

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1. (a) Prove that every finite integral domain is a field. [ 25 marks]
  - (b) If  $R$  is a commutative ring with unity, show that an ideal  $M$  in  $R$  is maximal if and only if  $R/M$  is a field. [40 marks]
  - (c) Let  $R$  be a principal ideal domain. Prove that every non-zero non-unit element of  $R$  can be expressed as a product of irreducible elements. [35 marks]
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2. (a) If  $F$  is a field, prove that  $F[x]$  is a Euclidean domain. [40 marks]
  - (b) State and prove Eisenstein's criterion. [25 marks]
  - (c) Let  $R$  be a unique factorization domain [ufd]. Show that the polynomial ring  $R[x]$  is also a ufd. [35 marks]

3. (a) Prove that an ideal  $\langle p \rangle$  in a principal ideal domain is maximal if and only if  $p$  is irreducible. [40 marks]
- (b) Show that the set  $\mathbb{Z}[i]$  of Gaussian integers is a Euclidean domain. [30 marks]
- (c) Show, by an example, that every integral domain need not be a unique factorization domain. [30 marks]
4. (a) In a ring  $\mathbb{Z}$  of integers, let  $p$  be a prime integer and let that, for some integer  $c$ , relatively prime to  $p$ , there exist integers  $x$  and  $y$  such that  $x^2 + y^2 = cp$ . Prove that integers  $a$  and  $b$  be found such that  $p = a^2 + b^2$ . [40 marks]
- (b) Let  $M$  be an  $R$ -module and  $x \in M$ . Show that the set  $K = \{ rx + nx \mid r \in R, n \in \mathbb{Z} \}$  is an  $R$ -submodule of  $M$  containing  $x$ . [30 marks]
- (c) Prove that the submodules of the quotient module  $M/N$  are of the form  $U/N$  where  $U$  is a submodule of  $M$  containing  $N$ . [30 marks]
5. (a) Let  $R$  be a ring with unity and  $\text{Hom}_R(R, R)$  denote the ring of endomorphisms of  $R$  regarded as a right  $R$ -module. Prove that  $R \simeq \text{Hom}_R(R, R)$  as rings. [30 marks]
- (b) Let  $M$  be a finitely generated free module over a commutative ring  $R$ . Show that all bases of  $M$  have the same number of generators. [30 marks]
- (c) Prove, for an  $R$ -module  $M$ , that the following are equivalent:
- $M$  is noetherian;



- ii. Every submodule of  $M$  is finitely generated;
- iii. Every nonempty set  $S$  of submodules of  $M$  has a maximal element.

[40 marks]

FIRST SEMESTER

6. (a) Let  $R$  be a principal ideal domain and  $M$  be a free module of rank  $n$  over  $R$  and  $N$  be a submodule of  $M$ . Prove that  $N$  is a free module of rank  $\leq n$ .

[45 marks]

- (b) Let  $M$  be a finitely generated module over a principal ideal domain. Prove that  $M$  can be expressed as  $M = F \oplus t(M)$  where  $F$  is a free submodule of  $M$  and  $t(M)$  is the torsion submodule of  $M$ , and  $F$  is unique up to isomorphism.

[55 marks]