# EASTERN UNIVERSITY, SRI LANKA
## SPECIAL DEGREE EXAMINATION
## IN MATHEMATICS, (2001/2002)
## (Jan.' 2004) - PART II
## MT 408 - PARTIAL DIFFERENTIAL EQUATIONS

Answer **four** questions only | Time allowed: **3 Hours**

Q1. State the connection between the solutions of the first order quasi-linear partial differential equation

$$P(x,y,u)\,\frac{\partial u}{\partial x}+Q(x,y,u)\,\frac{\partial u}{\partial y}=R(x,y,u) \qquad (1)$$

and the solutions of the system

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{du}{R}.$$

Define the characteristic curves of the partial differential equation (1).

[20 Marks]

(i) Find the solution $u=u(x,y)$ of the partial differential equation

$$(x-y)\,\frac{\partial u}{\partial x}+(y-x-u)\,\frac{\partial u}{\partial y}=u$$

which passes through the circle

$$u=1, \quad x^2+y^2=1.$$

[40 Marks]

ii) The density $\rho(x,t)$ satisfies the partial differential equation

$$\frac{\partial \rho}{\partial t} + \rho\,\frac{\partial \rho}{\partial x} = 0,$$

subject to the initial condition

$$\rho(x,0) = f(x) = \begin{cases} 1 & x \leq 0 \\ 1 - x & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

Formulate the problem as a Cauchy initial value problem. Sketch the projections of the characteristic curves on the $(x,t)$ plane. Determine the point in the $(x,t)$ plane at which the characteristic projections first intersect.

[40 Marks]

Q2. Explain briefly how to calculate the breaking time of a wave by formulating the problem as a Cauchy initial value problem.

[10 Marks]

Consider a model of damped nonlinear waves described by the PDE

$$\frac{\partial \rho}{\partial t} + \rho\,\frac{\partial \rho}{\partial x} = -a\rho$$

where $a > 0$ is a constant, subject to the initial conditions at $t = 0$:

$$\rho(x,0) = f(x), \quad -\infty \leq x \leq \infty.$$

Show that

$$\rho = e^{-at} f\left(x - \frac{1}{a}\left(e^{at} - 1\right)\rho\right)$$

and discuss the possible breaking of the wave.

[90 Marks]

Q3. Explain the significance of the Monge cone at a point on an integral surface of the first order nonlinear partial differential equation

$$F(x, y, u, p, q) = 0$$

where

$$u = u(x, y), \quad p = \frac{\partial u}{\partial x}, \quad q = \frac{\partial u}{\partial y},$$

and find the equation of the Monge cone at the point $(x_0, y_0, u_0)$ of an integral surface of the partial differential equation

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = u.$$

[25 Marks]

Obtain the solution of the equation

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = u, \qquad (2)$$

for $u(x, y)$ subject to the initial condition

$$u(x, 0) = x^2.$$

Show also that for general initial data in parametric form

$$x = x_0(\tau), \quad y = y_0(\tau), \quad u = u_0(\tau)$$

a necessary condition for a real solution of (2) to exist is

$$\left(\frac{du_0}{d\tau}\right)^2 \geq 4u_0 \frac{dx_0}{d\tau}\frac{dy_0}{d\tau}.$$

[75 Marks]

Q4. State the condition for the second order partial differential equation

$$R(x, y)\frac{\partial^2 u}{\partial x^2} + S(x, y)\frac{\partial^2 u}{\partial x \partial y} + T(x, y)\frac{\partial^2 u}{\partial y^2} = F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

to be parabolic. [10 Marks]

Reduce the partial differential equation

$$y^2 \frac{\partial^2 u}{\partial x^2} - 2xy \frac{\partial^2 u}{\partial x \partial y} + x^2 \frac{\partial^2 u}{\partial y^2} = \frac{y^2}{x}\frac{\partial u}{\partial x} + \frac{x^2}{y}\frac{\partial u}{\partial y}$$

to canonical form in the domain $x > 0$, $y > 0$. [55 Marks]

Obtain the solution $u(x, y)$ which satisfies the boundary conditions

$$u = 0, \quad \frac{\partial u}{\partial y} = 2x^2 \text{ on the line } y = 1.$$

[35 Marks]

Q5. Define the adjoint operator $L^*$ of the operator

$$L[u] = u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u.$$

[10 Marks]

By looking for a solution of the adjoint equation

$$L^*[v] = 0$$

of the form

$$v(x, y) = \frac{x}{y} W(x, y)$$

show that the Riemann function of the operator

$$L[u] = u_{xy} - \frac{1}{y} u_x + \frac{1}{x} u_y - \frac{1}{xy} u$$

is

$$R(x, y; x_1, y_1) = \frac{xy_1}{yx_1}.$$

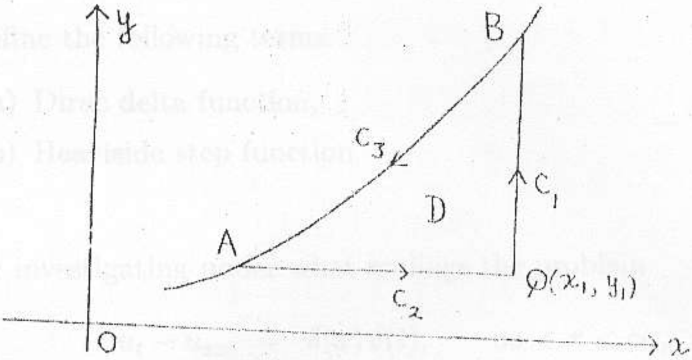
[50 Marks]

Hence solve the partial differential equation

$$u_{xy} - \frac{1}{y} u_x + \frac{1}{x} u_y - \frac{1}{xy} u = \frac{y}{x}$$

in the domain $x > 0$ and $y > 0$, subject to the initial conditions

$$u = x \sin x, \quad u_y = \sin x,$$




on the straight line $y = x$. [40 Marks]

Useful formulae:

$$u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y)$$

$$\begin{aligned} R(x_1, y_1; x_1, y_1) &= 1, \\ R(x_1, y; x_1, y_1) &= \exp\left(\int_{y_1}^{y} a(x_1, \sigma)d\sigma\right), \\ R(x, y_1; x_1, y_1) &= \exp\left(\int_{x_1}^{x} b(\sigma, y_1)d\sigma\right). \end{aligned}$$

$$\begin{aligned} u(x_1, y_1) = &\frac{1}{2}[R(A; x_1, y_1)\, u(A) + R(B; x_1, y_1)u(B)] \\ &-\int\int_D R(x, y; x_1, y_1)\, f(x, y)dxdy \\ &+\int_{c3}\left(auR + \frac{1}{2}Ru_y - \frac{1}{2}uR_y\right)dy - \left(buR + \frac{1}{2}Ru_x - \frac{1}{2}uR_x\right)dx. \end{aligned}$$

Q6. Define the following terms:

(a) Dirac delta function,

(b) Heaviside step function.

[10 Marks]

By investigating under what scalings the problem

$$\begin{aligned} u_t - u_{xx} &= \delta(x)\,\delta(t), \quad -\infty < x < \infty, \quad t \geq 0, \\ u(x, 0^-) &= 0, \quad -\infty < x < \infty, \end{aligned}$$

is invariant, show that its solution is of the form

$$u(x,t) = \frac{1}{\sqrt{t}}\, f\left(\frac{x^2}{t}\right).$$

[40 Marks]

You may use without proof any result for the Dirac delta function. Denote the solution of the problem

$$\begin{aligned} u_t - u_{xx} &= \delta(x-\xi)\,\delta(t-\tau), \quad -\infty < x < \infty, \quad t \geq \tau, \\ u(x, \tau^-) &= 0, \quad -\infty < x < \infty, \end{aligned}$$

by

$$u = F(x-\xi, t-\tau).$$

Show that the solution of the problem

$$\begin{aligned} u_t - u_{xx} &= p(x,t), \quad -\infty < x < \infty, \quad t \geq 0, \\ u(x, 0^-) &= 0, \quad -\infty < x < \infty, \end{aligned}$$

is

$$u(x,t) = \int_{\xi=-\infty}^{\infty} \int_{\tau=0^-}^{t} p(\xi,\tau)\, F(x-\xi, t-\tau)\, d\tau\, d\xi.$$

[50 Marks]