EASTERN UNIVERSITY, SRI LANKA

PECIAL DEGREE EXAMINATION IN MATHEMATICS

(2001/2002) PART II(Jan./Feb.2004)

MT 309-NUMERICAL LINEAR ALGEBRA

may attempt as many questions as you wish, but marks will be given for the best JR answers only. Time allowed is **THREE** hours only. Each question carries **ONE NDRED** marks. The numbers beside the questions indicate the approximate marks can be gained from the corresponding parts of the questions.

(a) Define the term "positive definite" as applied to an $n \times n$ Hermitian matrix A. Prove that the determinant of each principal sub-matrix of a positive definite matrix is positive. [20]

(b) Define the term "elementary lower-triangular matrix".

Prove that a positive definite matrix can be uniquely expressed as $A = LU$, where L is a unit lower triangular matrix and U is an upper triangular matrix. [30]

(c) Show that a Hermitian matrix A is positive definite if, and only if $A = GG^H$, where G is a non-singular lower triangular matrix. [25]

Determine G such that

$$GG^H = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 4 & -2 \\ 0 & 0 & -2 & 3 \end{bmatrix}.$$ [25]

2. (a) Define the terms "unitary matrix" and "elementary Hermitian matrix ". [10]

(b) Show that, for any real vector x, there is a real elementary Hermitian matrix $H(w)$ such that $H(w)x = ce_1$, where $c = x^T x$ and $e_1 = (1, 0, 0, \ldots, 0)^T$.

What is the optimal choice of the sign of c for the computation of w? [20]

(c) Let $H(w)$ be an $n \times n$ elementary Hermitian matrix and let I be the $m \times m$ identity matrix. Show that the partitioned matrix

$$\left[\begin{array}{c|c} I & O \\ \hline O & H(w) \end{array}\right]$$

is an elementary Hermitian matrix. [20]

(d) Determine an upper triangular matrix U such that $HA = U$, where H is a product of elementary Hermitian matrices and

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 4 & -1 \\ 2 & 5 & 0 \end{bmatrix}$$

making the optimal choice of sign in each stage of process. Hence solve the system $Ax = b$, where $b = (5, 0, -1)^T$ [50]

3. (a) Define the phrase strictly diagonally dominant applied to an $n \times n$ matrix A. [05]

(b) Given that A is strictly diagonally dominant, prove that A is non-singular. Given also that $A = I - L - U$, where L is strictly lower triangular and U is strictly upper triangular. Verify that $\| L + U \|_\infty < 1$ and obtain a bound for $\| A^{-1} \|_\infty$. [25]

(c) For arbitrary $x^{(0)}$, a sequence $\{x^{(r)}\}$ is defined by

$$x^{(r+1)} = (I - wL)^{-1}\{wb + [(1-w)I + wU]x^{(r)}\}, \quad r = 0, 1, 2, 3 \ldots$$

Show that $x - x^{(r+1)} = M(x - x^{(r)})$, $r = 0, 1, 2, \ldots$, where $M = (I - wL)^{-1}[(1-w)I + wU]$ and $Ax = b$. State a necessary and sufficient condition for $\{x^{(r)}\}$ to converge to x. [10]

(d) Let $0 < w \leq 1$ and let λ be any complex number with $|\lambda| \geq 1$. Show that $|\lambda + w - 1| \geq |w\lambda| \geq w$. Deduce that if λ is any eigenvalue of M, then $|\lambda| < 1$. [30]

(e) The following equations are to be solved by successive over-relaxation with a relaxation parameter 1.1.

Starting with $x^{(0)} = 0$, obtain $x^{(1)}$, $x^{(2)}$ and bound for $\| x - x^{(2)} \|_\infty$.

$$\begin{bmatrix} 11 & 1 & 0 & 0 \\ 1 & 11 & 1 & 0 \\ 0 & 1 & 11 & 2 \\ 0 & 0 & 2 & 11 \end{bmatrix} x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

[30]

(a) Define the term "upper Hessenberg matrix." [05]

(b) i. Let A be an $n \times n$ matrix. Describe how a non-singular matrix S, a product of elementary lower triangular matrices and elementary permutation matrices, can be obtained so that $S^{-1}AS$ is an upper Hessenberg matrix. [30]

ii. Obtain the number of multiplications needed for this process. Explain why Householder's method is better than this process when A is Hermition. [30]

(c) Given

$$A = \begin{bmatrix} 2 & -1 & 2 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & -1 & 2 & 1 \\ -1 & 0 & -1 & 2 \end{bmatrix},$$

find an upper Hessenberg matrix $S^{-1}AS$, where S is a product of elementary permutation matrices and elementary lower triangular matrices. [35]

5. (a) Let $A = (a_{ij})$ be an $n \times n$ upper Hessenberg matrix such that $a_{21}a_{32}\ldots a_{nn-1} \neq 0$.

Show that the characteristic polynomial $p(\lambda)$ of A is given by

$$p(\lambda) = \alpha_{n+1}(\lambda)a_{21}a_{32}\ldots a_{n\ n-1},$$

where α_{n+1} is given by the recurrence relation

$$\alpha_r\lambda = \alpha_1 a_{1r} + \alpha_2 a_{2r} + \cdots + \alpha_r a_{rr} + \alpha_{r+1}a_{r+1r},\ r = 1, 2, \ldots, n,$$

α_r is a function of λ, $r = 1, 2, \ldots, n$.

(Assume that $\alpha_1 = 1$, $a_{n+1\ n} = 1$).

[35]

(b) The upper Hessenberg matrix

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

has one eigenvalue λ in $(3.4, 3.5)$. Find the characteristic polynomial of A, and its derivative at $\lambda_0 = 3.5$. Apply one step of the Newton method to obtain a new estimate for λ. [65]

(a) i. Suppose that the eigenvalue λ_1 of largest modulus and corresponding eigenvector z_1 of an $n \times n$ matrix A have been computed by the Power method. Show that there is a non-singular matrix S, a product of an elementary permutation matrix and an elementary lower triangular matrix, such that

$$A = S \left[\begin{array}{c|c} \lambda_1 & \gamma^T \\ \hline O & B \end{array} \right] S^{-1},$$

where B is an $(n-1) \times (n-1)$ matrix and γ is an $(n-1)$-column vector. [25]

ii. Describe how the other eigenvalues and eigenvectors of A could be computed. [20]

(b) It is given that the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

has an eigenvalue close to 3.4 and that a corresponding eigenvector approximately $(0.7, 1, 0.3)^T$. Obtain 2×2 matrix B whose eigenvalues approximate the other eigenvalues of A . [30]

(c) Describe briefly how inverse iteration is used to improve an approximate eigenvalue and eigenvector of an $n \times n$ matrix. [25]