EASTERN UNIVERSITY, SRI LANKA

SPECIAL DEGREE EXAMINATION IN MATHEMATIC

2001/2002 (Jan.'2004)

FIRST SEMESTER

MT 402 - MEASURE THEORY

Answer all questions

Time : Three hours

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1. (a) Let (ϕ_n) be an increasing sequence of step functions on $\mathbb{R}$ such that $(\int \phi_n)$ is convergent. Prove from first principle that (ϕ_n) converges almost everywhere on $\mathbb{R}$.

   (b) Show that a subset E of $\mathbb{R}$ is null if, and only if, there is an increasing sequence (ϕ_n) of step functions on $\mathbb{R}$ such that

   i. $\lim_{n\to\infty} \int \phi_n < \infty$ and

   ii. for all $x \in \mathbb{E}$, $\phi_n(x) \to \infty$ as $n \to \infty$.

2. (a) Let $f \in L^1(\mathbb{R})$. Prove that there exists a sequence (ϕ_n) of step functions such that $\phi_n(x) \to f(x)$ almost everywhere in $\mathbb{R}$, and that

$$\int |f - \phi_n| \to 0 \quad \text{as} \quad n \to \infty.$$

(b) Prove that the function $x \to f(x) \cos kx$ belongs to $L^1(\mathbb{R})$ for each $k \in \mathbb{R}$, and that

$$\lim_{k\to\infty} \int_{\mathbb{R}} f(x) \cos kx \, dx = 0.$$

3. (a) Let f be a function which vanishes outside the interval $[a, b]$. Prove that if f is bounded and if the points of discontinuity of f form a null set then $f \in L^1(\mathbb{R})$.

(b) State the Monotone convergence Theorem in $L^1(\mathbb{R})$ and use it to prove

i. $\displaystyle\int_{-\infty}^{\infty} e^{-|x|} \, dx = 2$, and

ii. $\displaystyle\lim_{n\to\infty} \int_0^1 \frac{1+x}{1+x^n} \, dx = \frac{3}{2}$.

4. State and prove the Dominated convergence theorem in $L^1(\mathbb{R})$. (Monotone convergence theorem may be assumed.)

Prove that, for $\beta > -1$,

$$\int_0^{\infty} \frac{e^{-\beta x}}{e^x + e^{-x}} \, dx = \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1+\beta)}.$$

5. State Fubini's Theorem and Tonnelli's theorem for Lebesgue integrable function on $\mathbb{R}^2$.

(a) Prove that

$$f(x, y) = (x - \sin x)y^2 e^{-xy} \in L^1\left((0, \infty) \times (0, \infty)\right)$$

and deduce that

$$\int_0^\infty \frac{x - \sin x}{x^3}\, dx = \frac{\pi}{4}.$$

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\begin{aligned} f(x, y) &= \frac{x^2 - y^2}{(x^2 + y^2)^2} \quad \text{if} \quad (x, y) \neq (0, 0) \\ &= \quad 0 \qquad\qquad \text{if} \quad (x, y) = (0, 0). \end{aligned}$$

Does $f \in L^1((0, 1) \times (0, 1))$? Justify your answer.

6. Define the term measurable function.

(a) Prove that $f : \mathbb{R} \to \mathbb{R}$ is measurable if, and only if $\mathrm{mid}(-g, f, g)$ is integrable whenever g is a non-negative integrable function.

(b) Let (f_n) be a sequence of measurable functions on $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ and $f_n \to f$ almost everywhere. Prove that f is measurable.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function and let $c \in \mathbb{R}$. Prove that $\{x \in \mathbb{R} \mid f(x) \leq c\}$ is measurable.
(State without proof, any convergence theorem that you use.)