# SPECIAL DEGREE EXAMINATION IN MATHEMATICS

# 2004/2005 (March/April, 2007)

## MT 402- MEASURE THEORY

#### Part II

Answer all questions

### Time:Three Hours

This paper consists of 6 questions in a total of 3 pages



- 1. (a) Let  $A \subseteq \mathbb{R}$ , with  $m^*(A) < \infty$ . Prove that the following four statements are equivalent:
  - i. A is measurable;
  - ii.  $\forall \epsilon > 0, \exists \text{ open } U \supseteq A \text{ with } m^*(U \setminus A) < \epsilon;$
  - iii.  $\exists G \in G_{\delta}$  with  $A \subseteq G$  and  $m^{\star}(G \setminus A) = 0$ ;
  - iv.  $\forall \epsilon > 0 \ \exists B$ , a finite union of open (finite)intervals so that  $m^*(A \triangle B) < \epsilon$ .
  - (b) Let A be a measurable subset of  $\mathbb{R}$ , with m(A) > 0. Prove that  $m(A+x) = m(A), \forall x \in \mathbb{R}$ ;
  - ii. there exists a non-measurable subset P of [0,1);
  - iii. If  $A^* = \{x y : x, y \in A\}$ , then  $A^*$  contains an interval  $[-\alpha, \alpha]$  for some  $\alpha > 0$ .
- 2. Prove that
  - (a) if  $\{A_n\}_{n=1}^{\infty}$  is an increasing infinite sequence of measurable sets in  $\mathbb{R}$ , then  $m\left(\bigcup_{n=1}^{\infty}A_n\right)=\lim_{n\to\infty}m(A_n);$
  - (b) if  $\{A_n\}_{n=1}^{\infty}$  is a decreasing infinite sequence of measurable sets in  $\mathbb{R}$  such that  $m(A_k) < \infty$  for some k, then  $m\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} m(A_n)$ ;
  - (c) if the condition  $m(A) < \infty$  is dropped part (b) fails to hold;
  - (d) Let A be a measurable subset of  $\mathbb R$  with  $m(A)<\infty$ , then the function  $x\mapsto m\,(A\cap(-\infty,x])$  is continuous.

- 3. Let  $(X, \mathcal{B}, \mu)$  be a measure space.
  - (a) What does it to mean to say that a function  $f: X \to (-\infty, \infty)$  is  $\mathcal{B}$  measurable?
  - (b) Prove that if  $\mathbb{F}$  is a countable, non-void set of such functions f and if  $g(x) = \sup\{f(x) : f \in \mathbb{F}\}\$  for each  $x \in X$ , then g is  $\mathcal{B}$  measurable.
  - (c) Give an example of  $X, \mathcal{B}$ , and  $\mathbb{F}$  to show that the assertion in part (b) can fail if "countable" is omitted.
  - (d) Let g be an integrable function over a measurable set  $A \subseteq \mathbb{R}$ . Let  $\{f_n\}$  be a sequence of measurable functions such that  $|f_n(x)| \leq g(x) \ \forall x \in A$  and  $\lim_{n \to \infty} f_n(x) = f(x)$  a.e on A. Prove that  $\int_A f = \lim_{n \to \infty} \int_A f_n$ .

    Deduce that  $\lim_{n \to \infty} \int_a^\infty \frac{n^2 x e^{-n^2 x^2} dx}{1 + x^2} = 0$ , if a > 0. but, the result does not hold if a = 0
- 4. (a) Let  $(X, \Sigma, \mu)$  be a measure space, and the completion  $(X', \Sigma', \mu')$  of  $(X, \Sigma, \mu)$  be defined by  $\Sigma' = \{A \cup B | A \in \Sigma, B \subseteq C \text{ for some } C \in \Sigma, \mu(C) = 0\}$  and  $\mu'(A') = \mu(A)$  when  $A' = A \cup B$ .

  Prove that  $(X', \Sigma', \mu')$  is complete measure space.
  - (b) Let  $(X, \mathcal{B}, \mu)$  be a complete measure space. Let  $1 and <math>\mathcal{L}^p(X, \mathcal{B}, \mu)$  comprises all  $\mathcal{B}$  measurable functions f on X for which

$$\int_{X} |f|^{p} d\mu < \infty, \text{ and } ||f||_{p} = \left(\int_{X} |f|^{p} d\mu\right)^{\frac{1}{p}} \text{ for } f \in \mathcal{L}^{p}(X, \mathcal{B}, \mu).$$
 Prove that

- i. if  $f, g \in \mathcal{L}^p(X, \mathcal{B}, \mu)$ , then  $f + g \in \mathcal{L}^p(X, \mathcal{B}, \mu)$  and  $||f + g||_p \le ||f||_p + ||g||_p$ ;
  - ii.  $\mathcal{L}^{p}(X,\mathcal{B},\mu)$  with  $||.||_{p}$  is a complete normed linear space.

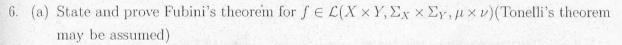
- 5. Let  $(X, \mathcal{B}, \mu)$  be a measure space. Prove that
  - (a) of  $\{f_n\}_{n=1}^{\infty}$  is an increasing sequence of non-negative measurable functions on X, with  $\lim_{n\to\infty} f_n = f$ , then  $\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu$ , but, the result does not hold for decreasing sequences.
  - (b) if  $\{f_n\}_{n=1}^{\infty}$  is a sequence of non-negative measurable functions on X, with  $\lim_{n\to\infty} f_n = f$  a. e, then  $\int_X f d\mu \leq \lim_{n\to\infty} \inf \int_X f_n d\mu$ , however strict inequality may not hold.
  - (c) if f, g are two non-negative measurable functions on X, and let a, b be non-negative constants, then af + bg is measurable and  $\int_{Y} (af + bg) d\mu = a \int_{Y} f d\mu + b \int_{Y} g d\mu$
  - (d) if  $\{f_n\}_{n=1}^{\infty}$  is a sequence of non-negative measurable functions on X, then

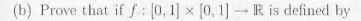
$$\int_{X} \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int_{X} f_n d\mu$$

(e) If

$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } 0 < x < \infty, \\ 1, & \text{if } x = 0 \end{cases}$$

then  $L \int_0^\infty f d\mu$  does not exist.





$$f(x,y) = \begin{cases} \frac{x^2 - y^2}{(x^2 + y^2)^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

then the iterated integrals are not equal. Is f integrable?

(c) By considering 
$$\int_0^a \int_0^\infty e^{-xt} \sin x dt dx$$
 prove that 
$$\lim_{a \to \infty} \int_0^a \frac{\sin x}{x} dx = \frac{\pi}{2}$$

