EASTERN UNIVERSITY, SRI LANKA

SPECIAL DEGREE EXAMINATION

IN MATHEMATICS, (2004/2005)

(MARCH/APRIL, 2007)

PART II

MT 404 - PARTIAL DIFFERENTIAL EQUATIONS

Answer **all** questions | Time allowed: **3 Hours**

Q1. (i) State the connection between the solutions of the first- order quasi-linear partial differential equation

$$P(x,y,u)\,\frac{\partial u}{\partial x}+Q(x,y,u)\,\frac{\partial u}{\partial y}=R(x,y,u) \qquad (1)$$

and the solutions of the system

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{du}{R}.$$

Define the characteristic curves of the partial differential equation (1).

[20 Marks]

(ii) Find the solution $u=u(x,y)$ of the partial differential equation

$$x(y-u)\frac{\partial u}{\partial x}+y(u-x)\frac{\partial u}{\partial y}=u(x-y)$$

which passes through the curve

$$x=y=u.$$

[80 Marks]

Q2. (i) The density $\rho(x,t)$ satisfies the partial differential equation

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial \rho}{\partial x} = 0,$$

subject to the initial condition

$$\rho(x,0) = f(x) = \begin{cases} 1 & x \leq 0 \\ 1-x & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}.$$

Formulate the problem as a Cauchy initial value problem. Sketch the projections of the characteristic curves on the (x,t) plane. Determine the point in the (x,t) plane at which the characteristic projections first intersect. **[50 Marks]**

(ii) Explain briefly how to calculate the breaking time of a wave by formulating the problem as a Cauchy initial value problem.

[10 Marks]

Consider a model of damped nonlinear waves described by the PDE

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial \rho}{\partial x} = -a\rho$$

where $a > 0$ is a constant, subject to the initial condition at $t = 0$:

$$\rho(x,0) = f(x), \quad -\infty \leq x \leq \infty.$$

Show that

$$\rho = e^{-at} f\left(x - \frac{1}{a}\left(e^{at} - 1\right)\rho\right)$$

and discuss the possible breaking of the wave. **[40 Marks]**

Q3. Explain the significance of the Monge cone at a point on an integral surface of the first order nonlinear partial differential equation

$$F(x,y,u,p,q) = 0$$

where

$$u = u(x,y), \quad p = \frac{\partial u}{\partial x}, \quad q = \frac{\partial u}{\partial y},$$

and find the equation of the Monge cone at the point (x_0, y_0, u_0) of an integral surface of the partial differential equation

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = u.$$

[25 Marks]

Find the two solutions of the partial differential equation

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = u \tag{2}$$

which satisfy the initial conditions

$$u = 1 + \frac{1}{4}x^2 \quad \text{on} \quad y = 0.$$

Comment briefly on the non-uniqueness of the solution to this problem. Show also that for general initial conditions in parametric form

$$x = x_0(\tau), \quad y = y_0(\tau), \quad u = u_0(\tau),$$

a necessary condition for a real solution of (2) to exist is

$$\left(\frac{du_0}{d\tau}\right)^2 \leq u_0(\tau)\left[\left(\frac{dx_0}{d\tau}\right)^2 + \left(\frac{dy_0}{d\tau}\right)^2\right].$$

You may assume that either $\frac{dx_0}{d\tau} \neq 0$ or $\frac{dy_0}{d\tau} \neq 0$. [75 Marks]

Q4. State the condition for the second order partial differential equation

$$R(x,y)\frac{\partial^2 u}{\partial x^2} + S(x,y)\frac{\partial^2 u}{\partial x \partial y} + T(x,y)\frac{\partial^2 u}{\partial y^2} = F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

to be hyperbolic. [10 Marks]

Show that, if $x \neq 0$ and $y \neq 0$, the partial differential equation

$$2x^2\frac{\partial^2 u}{\partial x^2} + 5xy\frac{\partial^2 u}{\partial x \partial y} + 2y^2\frac{\partial^2 u}{\partial y^2} + 8x\frac{\partial u}{\partial x} + 5y\frac{\partial u}{\partial y} = 0$$

is hyperbolic and that characteristic coordinates are

$$\xi = \frac{x^2}{y}, \quad \eta = \frac{y^2}{x}.$$

Show that the canonical form is

$$\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{1}{\eta}\frac{\partial u}{\partial \xi} = 0.$$

Obtain the general solution for $u(x, y)$. [55 Marks]

Hence obtain the particular solution $u(x, y)$ which satisfies the boundary conditions

$$u(1, y) = y^2, \quad \frac{\partial u}{\partial y}(1, y) = 1.$$

[35 Marks]

Q5. Define the adjoint operator L^* of the operator

$$L[u] = u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u.$$

[10 Marks]

By looking for a solution of the adjoint equation

$$L^*[v] = 0$$

of the form

$$v(x, y) = (x + y)\, W(x + y),$$

show that the Riemann function of the operator

$$L[u] = u_{xy} + \frac{1}{x+y}\, u_x + \frac{1}{x+y} u_y$$

is

$$R(x, y; x_1, y_1) = \frac{x + y}{x_1 + y_1}.$$

[50 Marks]

Hence solve the partial differential equation

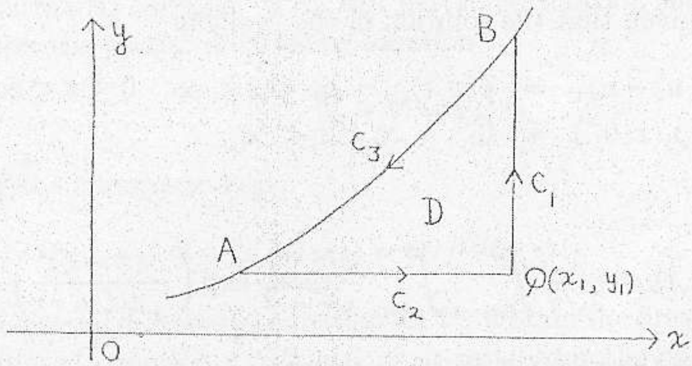
$$u_{xy} + \frac{1}{x+y}\, u_x + \frac{1}{x+y} u_y = \frac{1}{x+y}$$

subject to the initial conditions

$$u = x,\ u_y = 1 - x^2,$$

on the straight line $y = x$. [40 Marks]

Useful formulae:



$$u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y).$$

$$\begin{aligned} R(x_1, y_1; x_1, y_1) &= 1, \\ R(x_1, y; x_1, y_1) &= \exp\left(\int_{y_1}^{y} a(x_1, \sigma)d\sigma\right), \\ R(x, y_1; x_1, y_1) &= \exp\left(\int_{x_1}^{x} b(\sigma, y_1)d\sigma\right). \end{aligned}$$

$$\begin{aligned} u(x_1, y_1) = \frac{1}{2}[R(A; x_1, y_1)\, u(A) + R(B; x_1, y_1)u(B)] \\ - \int\int_D R(x, y; x_1, y_1)\, f(x, y)dxdy \\ + \int_{c3}\left(auR + \frac{1}{2}Ru_y - \frac{1}{2}uR_y\right)dy - \left(buR + \frac{1}{2}Ru_x - \frac{1}{2}uR_x\right)dx. \end{aligned}$$

Q6. Define the following terms:

(a) Dirac delta function,

(b) Heaviside step function.

[10 Marks]

You are given that the solution of the problem

$$\begin{aligned} u_t - u_{xx} &= p(x,t), \quad -\infty < x < \infty, \quad 0 \le t < \infty, \\ u(x,0^-) &= 0, \quad -\infty < x < \infty, \end{aligned}$$

is

$$u(x,t) = \frac{1}{2\sqrt{\pi}} \int_{\xi=-\infty}^{\infty} \int_{\tau=0^-}^{t} \frac{p(\xi,\tau)}{\sqrt{t-\tau}} \exp\left(-\frac{(x-\xi)^2}{4(t-\tau)}\right) d\tau\, d\xi.$$

Show that the solution of the problem

$$\begin{aligned} u_t - u_{xx} &= 0, \quad -\infty < x < \infty, \quad 0 \le t < \infty, \\ u(x,0^+) &= f(x), \quad -\infty < x < \infty, \end{aligned}$$

is

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\xi) \exp\left(-\frac{(x-\xi)^2}{4t}\right) d\xi.$$

[40 Marks]

A certain financial option $V(s,t)$ satisfies the partial differential equation

$$V_t = s^2 V_{ss} + sV_s - V, \quad 0 \le s < \infty, \quad 0 \le t < \infty$$

and the initial condition

$$V(s,0^+) = \begin{cases} 0, & 0 < s < E \\ 1, & E < s < \infty \end{cases}$$

where $E > 0$ is a positive constant.

Make the change of variables

$$x = f(s)$$

in the partial differential equation for V. Obtain the second-order ordinary differential equation which $f(s)$ must satisfy for the coefficient of V_x to vanish. Obtain the general solution for $f(s)$ and verify that

$$f(s) = \ln s$$

is a particular solution. With this particular solution show that $V(x,t)$ satisfies the partial differential equation

$$V_t + V = V_{xx}.$$

Make the transformation

$$V(x,t) = g(t)W(x,t)$$

in the partial differential equation for V. Obtain the first-order ordinary differential equation which $g(t)$ must satisfy for the coeffient of W to vanish. Obtain the general solution for $g(t)$ and verify that

$$g(t) = e^{-t}$$

is a particular solution. With this particular solution write down the partial differential equation satisfied by $W(x,t)$. If

$$V(x,t) = e^{-t}W(x,t)$$

where

$$x = \ln s,$$

show that $W(x,t)$ satisfies the problem

$$\begin{aligned} W_t - W_{xx} &= 0, \quad -\infty < x < \infty, \quad 0 \leq t < \infty, \\ W(x,0^+) &= H(x - \ln E), \quad -\infty < x < \infty, \end{aligned}$$

where H is the Heaviside step function. Deduce that

$$V(s,t) = \frac{e^{-t}}{2\sqrt{\pi t}} \int_E^\infty \exp\left(-\frac{(\ln(s/\sigma))^2}{4t}\right) \frac{d\sigma}{\sigma}.$$

[50 Marks]