EASTERN UNIVERSITY, SRI LANKA
SPECIAL DEGREE EXAMINATION IN MATHEMATICS (2004/2005)
MARCH/APRIL, 2007
PART I
MT 406 – APPROXIMATION THEORY

Time: 3 Hours **Maximum Marks: 600**

Answer ALL Questions

I. (*a*) If K is a closed convex subset of R^n, then show that K possesses a unique point of minimum norm.

(*b*) Show, if X is a uniformly convex Banach space and $K \subset X$ is a closed convex set, that each $f \in X$ has a unique best approximation p^* from K.

(*c*) Let X be a strictly convex normed space and $M \subset X$ be a finite dimensional subspace. Prove that each $f \in X$ has a unique best approximation from M. [$25 + 40 + 35 = 100$]

II. (*a*) Let $f \in C[a,b]$ and let $g_1, \ldots, g_n \in C[a,b]$ with $g_1, \ldots, g_n$ linearly independent. Define $\hat{x} = (g_1(x), \ldots, g_n(x))$, $x \in [a,b]$. Prove that for $P = \Sigma c_i g_i$ to be a best approximation, that is $c_1, c_2, \ldots, c_n$ to be such that the residual $r = f - \sum_{i=1}^{n} c_i g_i$ has minimum norm, it is necessary and sufficient that $\underline{0} \in Co\{r(x)\hat{x} : x \in [a,b] \text{ and } |r(x)| = \|r\|\}$.

(*b*) Let $\{g_1, g_2, \ldots, g_n\}$ form a Chebyshev system on $[a,b]$. Let $a \le x_0 < x_1 < x_2 < \ldots < x_n < b$ and $\lambda_0, \lambda_1, \ldots, \lambda_n \neq 0$. Prove that in order that $\underline{0} \in Co\{\lambda_0 \hat{x}_0, \lambda_1 \hat{x}_1, \ldots, \lambda_n \hat{x}_n\}$, it is necessary and sufficient that $\lambda_j \lambda_{j+1} < 0$, $j = 0,1,2,\ldots, n-1$. [$55 + 45 = 100$]

III. (*a*) Prove: $\min_{c_1, c_2, \ldots, c_{n-1}} \int_0^{\pi} \left| x - \sum_{k=1}^{n-1} c_k \sin(kx) \right| dx = \pi^2/(2n)$.

(*b*) Define the modulus of continuity of $f \in C_{2\pi}$ and, for $f \in C_{2\pi}$, prove that

$$\mathcal{E}_n[f] \le (3/2)\, \omega(f; \tfrac{\pi}{n+1}),\ n = 1,2,3,\ldots$$

(*c*) Let $f \in C_{2\pi}$ and $0 < \alpha < 1$. Prove that f satisfies the condition that, for some $B > 0$, $|f(x) - f(y)| \le B|x - y|^{\alpha}$, for all $x,y \in [0,2\pi]$ if there exists $A > 0$ such that $\mathcal{E}_n[f] \le An^{-\alpha}$, $n \ge 1$. [$30 + 25 + 45 = 100$]

IV. (*a*) Let $f \in C[-1,1]$ and let k be a positive integer and let $0 < \alpha < 1$. Assume that, for some $A > 0$, $\mathcal{E}_n[f] \le An^{-k-\alpha}$, $n \ge 1$. Show that $f^{(k)}$ exists and is continuous in $(-1,1)$ and, given $0 < \delta < 1$, there exists $B > 0$ such that $|f^{(k)}(x) - f^{(k)}(y)| \le B|x - y|^{\alpha}$, for all $x,y \in [-1 + \delta, 1 - \delta]$.

(*b*) Let X be the space of continuous functions $f: [0,1] \rightarrow R$ with inner product $(f,g) = \int_0^1 f(x)g(x)dx$. Let M be a finite dimensional subspace of X with basis $\{x^{\alpha_1}, x^{\alpha_2}, \ldots, x^{\alpha_n}\}$. $\alpha_1, \alpha_2, \ldots, \alpha_n \ge 0$, distinct. Prove that the distance from x^m $(m \ge 0)$

to M is $d = \dfrac{1}{\sqrt{2m+1}} \prod_{j=1}^{n} \left| \dfrac{m-\alpha_j}{m+\alpha_j+1} \right|$.

(*c*) Let X be the inner product space of continuous functions $f: [0,1] \to R$ with inner product $(f,g) = \int_0^1 f(x)g(x)dx$, and norm induced by the inner product. Let $\alpha_1, \alpha_2, \alpha_3, \ldots$ be distinct non-negative numbers. Show that $\mathcal{A} = \{x^{\alpha_1}, x^{\alpha_2}, \ldots\}$ is fundamental in X if and only if $\sum_{j=1}^{\infty} 1/\alpha_j = \infty$. [*35* + *25* + *40* = **100**]

V. (*a*) Let $\lambda = \int_0^\infty \log\left|\dfrac{t-1}{t+1}\right| \dfrac{dt}{t}$. Show, for all $b \geq a \geq 0$ and $z \in C$, that $\int_a^b \log\left|\dfrac{t+z}{t-z}\right| \dfrac{dt}{t} \geq \lambda$.

(*b*) Let $f(x) = |x|$, $x \in [-1,1]$. Then prove that there exists C_1 such that $\frac{1}{2} e^{C_1\sqrt{n}} \leq r_n(f) \leq 8e^{-\sqrt{n}/5}$, $n \geq 36$. [*50* + *50* = **100**]

VI. (*a*) Let $r > 1$ and f be analytic inside the ellipse $\mathcal{E}_r = \{z = \varphi(\omega) = (1/2)(\omega + 1/\omega) : |\omega| = r\}$. For $n \geq 1$, let P_n be the Lagrange interpolation polynomial of deg $\leq n-1$ to f at $x_{1n}, \ldots, x_{nn}$, the zeros of T_n so that $P_n(x_{jn}) = f(x_{jn})$, $1 \leq j \leq n$. Let $1 < s < r$. Then prove that there exists $C > 0$ such that $\|f - P_n\|_{[-1,1]} \leq C/s^n$, $n \geq 1$.

(*b*) If P is a polynomial of deg $\leq n$, show that $|P(\omega)| \leq |\omega|^n \max_{|t|=1} |P(t)|$, $|\omega| \geq 1$.

(*c*) Let $f \in C[-1,1]$ and assume that, for some $r > 1$, $\limsup_{n\to\infty} E_n[f]^{1/n} \leq 1/r$. Show that f is the restriction to $[-1,1]$ of a function analytic inside $\mathcal{E}_r$. [*30* + *25* + *45*(*20*+*25*) = **100**]