EASTERN UNIVERSITY, SRI LANKA

SPECIAL DEGREE EXAMINATION

IN MATHEMATICS, (2004/2005)

(MARCH/APRIL, 2007)

PART I

MT 408 - RELATIVITY

Answer all questions Time allowed: **3 Hours**

Q1. (a) A number of point charges $e_1, e_2, \ldots, e_N$ are fixed at interior points $Q_1, Q_2, \ldots, Q_N$ of a line OX. Show that if P is a point on any selected line of force, then

$$\sum_{i=1}^{N} e_i \cos\theta_i = \text{constant, where } \theta_i = P\hat{Q}_i X.$$

(b) Positive charges q_1 and q_2 are placed at points A and B respectively. Consider the line of force starting from A at an angle α to BA. Prove that its asymptote passes through the point C on AB such that $\frac{AC}{CB} = \frac{q_2}{q_1}$ and makes and angle β with BA given by

$$\sin\left(\frac{\beta}{2}\right) = \left(\frac{q_1}{q_1+q_2}\right)^{\frac{1}{2}} \sin\left(\frac{\alpha}{2}\right).$$

(c) Show that the electric field potential due to dipole $\mathbf{P}$ is

$$\frac{1}{4\pi\epsilon_0}\frac{\mathbf{P}.\mathbf{r}}{r^3},$$

where $\mathbf{r}$ is the vector from a dipole to the point concerned.

Q2. In spherical coordinates, the Laplace equation is given by:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial v}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2} = 0.$$

(a) Show that the general solution of the Laplace equation in the axially symmetric case is given by:

$$v(r,\theta) = \sum_{n=0}^{\infty} \left(C_n r^n + D_n r^{-(n+1)}\right) P_n(\cos\theta),$$

where $P_n(x)$ is the Legendre polynomial of degree n.

(b) An earthed conducting sphere of radius a is coated with a thickness $b-a$ of dielectric of dielectric constant k. The sphere and dielectric are placed in a uniform electric field E.

(i) Show that the change in the field outside the dielectric is the same as that produced by an electric dipole of moment

$$4\pi\epsilon_0 E b^3 \left(\frac{(2k+1)a^3 + (k-1)b^3}{2(k-1)a^3 + (k+2)b^3}\right)$$

at the centre of the sphere.

(ii) Show also that the surface density of charge at a point on the conductor is

$$\frac{qk\epsilon_0 E\cos\theta}{k+2+2(k-1)\frac{a^3}{b^3}}$$

where θ is the angle between the radius to the point and the direction of the field.

Q3. (a) Discuss the significance of

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\, dv'$$

in connection with Ampere's Law, giving the force between two current carrying loops of steady current.

(b) Prove that:

(i) curl $\mathbf{B} = \mu_0 \mathbf{j}$;

(ii) $\mathbf{B} =$ curl $\mathbf{A}$, where $\mathbf{A} = \dfrac{\mu_0}{4\pi}\displaystyle\int_V \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dv'$;

(iii) div $\mathbf{A} = 0$;

(iv) $\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\mathbf{m} \wedge \nabla\left(\frac{1}{r}\right)$ for a magnetic pole of strength $\mathbf{m}$ at the origin.

(v) volume distribution $\mathbf{m}(\mathbf{r})$ of magnetic poles, gives rise to a virtual current distribution $\nabla \wedge \mathbf{m}(\mathbf{r})$.

(c) Discuss the generalisation of curl $\mathbf{B} = \mu_0 \mathbf{j}$ to unsteady currents.

Q4. In two spacetime dimensions two observers moving with constant relative velocity v set up inertial frames $\mathcal{R}$ and $\mathcal{R}'$ with coordinate systems (ct, x) and (ct', x') respectively.

(a) Starting with a linear transform and the invariance of the speed of light, show that if they set their clocks to $t = t' = 0$ when they pass each other, the transform between these coordinate systems is the Lorentz transform:

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \gamma(v) \begin{pmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}, \quad \text{where } \gamma(v) = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}.$$

(b) For the following you should state any results you use but do not have to prove them:

(i) Show that a particle moving with the speed of light in $\mathcal{R}$ also moves at the speed of light in $\mathcal{R}'$.

(ii) Two events are simultaneous in $\mathcal{R}$, but occur one second apart in $\mathcal{R}'$. Calculate the velocity in terms of c of $\mathcal{R}'$ relative to $\mathcal{R}$ if the distance between the event is 10^6km in $\mathcal{R}$.

(iii) A vehicle travels with speed $0.1c$ in $\mathcal{R}'$. How fast is it travelling in $\mathcal{R}$??

(iv) An object which is stationary in $\mathcal{R}$ has length 2m. How long does it appear in $\mathcal{R}'$?

Q5. Two electrons with rest mass m_0 each have energy E in the centre-of-mass frame.

(a) Show that in the laboratory frame in which one electron is originally at rest, the other has initial energy $\left(2E^2 - m_0^2 c^4\right)/\left(m_0 c^2\right)$.

(b) The electrons collide elastically and then move at right-angles to their original directions, as measured in the centre-of-mass frame. Find the angle between the electrons after the collision as measured in the laboratory frame, and their new energy.

(c) The experiment is repeated, but this time after the collision there are two electrons and a π^0 meson (with rest mass m_0^π). What is the minimum velocity of the moving electron required for this to occur?

Q6. (a) (i) Define the term 4–vector. What is the 4–momentum P of a massless particle (such as a photon)?

(ii) Define the inner produce $g(X, Y)$ for two 4–vectors and hence show that P is null. Does this hold for massive particles (such as electrons)?

(b) Consider light emitted at an angle θ' in the rest frame $\mathcal{R}'$ of a source moving with speed v.

(i) Show that the light has an observed angle θ satisfying:

$$\tan\theta = \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} \frac{\sin\theta'}{\cos\theta' + \frac{v}{c}}.$$

(ii) Furthermore show that a photon with energy E' in the rest frame $\mathcal{R}'$ of the source has observed energy E, where

$$E = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} E' \left(1 + \frac{v}{c}\cos\theta'\right).$$

(iii) Show that if $\frac{v}{c}$ is close to unity then any forward shining light $(-\frac{\pi}{2} \leq \theta' \leq \frac{\pi}{2})$ is observed to be concentrated in a narrow cone whose semi-angle θ is given by $\sin\theta = \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}$.