

in Universi

EASTERN UNIVERSITY, SRI LANKA
SPECIAL DEGREE EXAMINATION
IN MATHEMATICS, (2001/2002)

(Jan.' 2004) - PART II

MT 408 - PARTIAL DIFFERENTIAL EQUATIONS

Answer **four** questions only

Time allowed: **3 Hours**

Q1. State the connection between the solutions of the first order quasi-linear partial differential equation

$$P(x, y, u) \frac{\partial u}{\partial x} + Q(x, y, u) \frac{\partial u}{\partial y} = R(x, y, u) \quad (1)$$

and the solutions of the system

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{du}{R}$$

Define the characteristic curves of the partial differential equation (1).

[20 Marks]

(i) Find the solution $u = u(x, y)$ of the partial differential equation

$$(x - y) \frac{\partial u}{\partial x} + (y - x - u) \frac{\partial u}{\partial y} = u$$

which passes through the circle

$$u = 1, \quad x^2 + y^2 = 1.$$

[40 Marks]

ii) The density $\rho(x, t)$ satisfies the partial differential equation

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial \rho}{\partial x} = 0,$$

subject to the initial condition

$$\rho(x, 0) = f(x) = \begin{cases} 1 & x \leq 0 \\ 1 - x & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

Formulate the problem as a Cauchy initial value problem. Sketch the projections of the characteristic curves on the (x, t) plane. Determine the point in the (x, t) plane at which the characteristic projections first intersect.

[40 Marks]

Q2. Explain briefly how to calculate the breaking time of a wave by formulating the problem as a Cauchy initial value problem.

[10 Marks]

Consider a model of damped nonlinear waves described by the PDE

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial \rho}{\partial x} = -a\rho$$

where $a > 0$ is a constant, subject to the initial conditions at $t = 0$:

$$\rho(x, 0) = f(x), \quad -\infty \leq x \leq \infty.$$

Show that

$$\rho = e^{-at} f \left(x - \frac{1}{a} (e^{at} - 1) \rho \right)$$

and discuss the possible breaking of the wave.

[90 Marks]

Q3. Explain the significance of the Monge cone at a point on an integral surface of the first order nonlinear partial differential equation

$$F(x, y, u, p, q) = 0$$

where

$$u = u(x, y), \quad p = \frac{\partial u}{\partial x}, \quad q = \frac{\partial u}{\partial y},$$

and find the equation of the Monge cone at the point (x_0, y_0, u_0) of an integral surface of the partial differential equation

$$\frac{\partial u}{\partial x} \frac{\partial u}{\partial y} = u.$$

[25 Marks]

Obtain the solution of the equation

$$\frac{\partial u}{\partial x} \frac{\partial u}{\partial y} = u, \quad (2)$$

for $u(x, y)$ subject to the initial condition

$$u(x, 0) = x^2.$$

Show also that for general initial data in parametric form

$$x = x_0(\tau), \quad y = y_0(\tau), \quad u = u_0(\tau)$$

a necessary condition for a real solution of (2) to exist is

$$\left(\frac{du_0}{d\tau} \right)^2 \geq 4u_0 \frac{dx_0}{d\tau} \frac{dy_0}{d\tau}.$$

[75 Marks]

Q4. State the condition for the second order partial differential equation

$$R(x, y) \frac{\partial^2 u}{\partial x^2} + S(x, y) \frac{\partial^2 u}{\partial x \partial y} + T(x, y) \frac{\partial^2 u}{\partial y^2} = F \left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right)$$

to be parabolic.

[10 Marks]

Reduce the partial differential equation

$$y^2 \frac{\partial^2 u}{\partial x^2} - 2xy \frac{\partial^2 u}{\partial x \partial y} + x^2 \frac{\partial^2 u}{\partial y^2} = \frac{y^2}{x} \frac{\partial u}{\partial x} + \frac{x^2}{y} \frac{\partial u}{\partial y}$$

to canonical form in the domain $x > 0, y > 0$. [55 Marks]

Obtain the solution $u(x, y)$ which satisfies the boundary conditions

$$u = 0, \quad \frac{\partial u}{\partial y} = 2x^2 \quad \text{on the line } y = 1.$$

[35 Marks]

Q5. Define the adjoint operator L^* of the operator

$$L[u] = u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u.$$

[10 Marks]

By looking for a solution of the adjoint equation

$$L^*[v] = 0$$

of the form

$$v(x, y) = \frac{x}{y} W(x, y)$$

show that the Riemann function of the operator

$$L[u] = u_{xy} - \frac{1}{y} u_x + \frac{1}{x} u_y - \frac{1}{xy} u$$

is

$$R(x, y; x_1, y_1) = \frac{xy_1}{yx_1}.$$

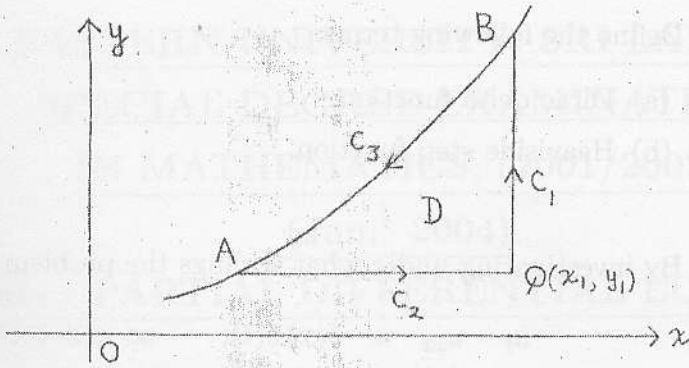
[50 Marks]

Hence solve the partial differential equation

$$u_{xy} - \frac{1}{y} u_x + \frac{1}{x} u_y - \frac{1}{xy} u = \frac{y}{x}$$

in the domain $x > 0$ and $y > 0$, subject to the initial conditions

$$u = x \sin x, \quad u_y = \sin x,$$



on the straight line $y = x$.

[40 Marks]

Useful formulae:

$$u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y).$$

$$R(x_1, y_1; x_1, y_1) = 1,$$

$$R(x_1, y; x_1, y_1) = \exp\left(\int_{y_1}^y a(x_1, \sigma) d\sigma\right),$$

$$R(x, y_1; x_1, y_1) = \exp\left(\int_{x_1}^x b(\sigma, y_1) d\sigma\right).$$

$$u(x_1, y_1) = \frac{1}{2}[R(A; x_1, y_1)u(A) + R(B; x_1, y_1)u(B)]$$

$$- \int \int_D R(x, y; x_1, y_1) f(x, y) dx dy$$

$$+ \int_{c_3} \left(auR + \frac{1}{2}Ru_y - \frac{1}{2}uR_y \right) dy - \left(buR + \frac{1}{2}Ru_x - \frac{1}{2}uR_x \right) dx.$$

Q6. Define the following terms:

- (a) Dirac delta function,
- (b) Heaviside step function.

[10 Marks]

By investigating under what scalings the problem

$$\begin{aligned}u_t - u_{xx} &= \delta(x) \delta(t), & -\infty < x < \infty, & t \geq 0, \\u(x, 0^-) &= 0, & -\infty < x < \infty,\end{aligned}$$

is invariant, show that its solution is of the form

$$u(x, t) = \frac{1}{\sqrt{t}} f\left(\frac{x^2}{t}\right).$$

[40 Marks]

You may use without proof any result for the Dirac delta function. Denote the solution of the problem

$$\begin{aligned}u_t - u_{xx} &= \delta(x - \xi) \delta(t - \tau), & -\infty < x < \infty, & t \geq \tau, \\u(x, \tau^-) &= 0, & -\infty < x < \infty,\end{aligned}$$

by

$$u = F(x - \xi, t - \tau).$$

Show that the solution of the problem

$$\begin{aligned}u_t - u_{xx} &= p(x, t), & -\infty < x < \infty, & t \geq 0, \\u(x, 0^-) &= 0, & -\infty < x < \infty,\end{aligned}$$

is

$$u(x, t) = \int_{\xi=-\infty}^{\infty} \int_{\tau=0^-}^t p(\xi, \tau) F(x - \xi, t - \tau) d\tau d\xi.$$

[50 Marks]