EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

SECOND EXAMINATION IN SCIENCE - 2008/2009

FIRST SEMESTER (Feb./Mar., 2010)

MT 201 - VECTOR SPACES AND MATRICES

Answer all questions Time: Three hours

1. Define the term *subspace* of a vector space.

(a) Let $V = \{f / \ f : \mathbb{R} \to \mathbb{R}, f(x) > 0, \forall\ x \in \mathbb{R}\}$. For any $f, g \in V$ and for any $\alpha \in \mathbb{R}$ define an addition $\oplus$ and a scalar multiplication $\odot$ as follows:

$$(f \oplus g)(x) = f(x).g(x), \forall x \in \mathbb{R}$$

and

$$(\alpha \odot f)(x) = (f(x))^{\alpha}.$$

Prove that $(V, \oplus, \odot)$ is a vector space over the set of real numbers $\mathbb{R}$.

(b) i. Let V be a vector space over a field $\mathbb{F}$. Prove that a non-empty subset W of V is a subspace of V if and only if $\alpha x + \beta y \in W$, for any $x, y \in W$ and $\alpha, \beta \in \mathbb{F}$.

ii. Let $P_n = \left\{ \sum_{i=0}^{n} a_i x^i \ : \ a_i \in \mathbb{R} \right\}$ be the set of all polynomials of degree $\leq n$ with real coefficients. Prove that P_n is a subspace of the vector space $V = \left\{ \sum_{i=0}^{n} a_i x^i \ : \ a_i \in \mathbb{R}, n \in \mathbb{N} \right\}$, the set of all polynomials with real coefficients.

Is it true that the set of polynomials exactly of degree n is a subspace of V? Justify your answer.

2. Define what is meant by *dimension* of a vector space.

   (a) Let V be an n-dimensional vector space.

   Show that

   i. a linearly independent set of vectors of V with n elements is a basis for V;

   ii. any linearly independent set of vectors of V may be extended to a basis for V;

   iii. if $\{e_1, e_2, \cdots, e_n\}$ is a basis for V then $V = \langle\{e_1, e_2, \cdots, e_r\}\rangle \oplus \langle\{e_{r+1}, e_{r+2}, \cdots$ $\forall\, r \in \{1, 2, \cdots, n-1\}$.

   (b) i. Let $w = (1, -1, 0, 3),\ x = (2, 1, 1, -1),\ y = (4, -1, 1, 3),\ z = (1, -4, -1, 8)$ be vectors in $\mathbb{R}^4$ and let $S = \langle\{w, x, y, z\}\rangle$. Find a basis and the dimension of S. Is the set $\{w, x, y, z\}$ linearly independent?

   Extend the basis of S that you obtained to a basis for $\mathbb{R}^4$.

   Find also a basis of $S \cap T$, where $T\ \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 + x_4 = 0\}$ is a subspace of $\mathbb{R}^4$.

   ii. Let $\{x_1, x_2, \cdots, x_n\}$ be any vectors in a vector space. Let $A = \langle\{x_1, x_2, \cdots$ and $B = \langle\{x_1, x_2, \cdots, x_n, x_{n+1}\}\rangle$. Prove that dim B = dim $A + \epsilon$, where is 0 if $x_{n+1} \in A$ and 1 if $x_{n+1} \notin A$.

3. (a) Let $D : P_n(t) \to P_{n-1}(t)$, the derivative operator, be defined by

$$D(p(t)) = a_1 + 2a_2 t + \cdots + n a_n t^{n-1}$$

   where $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$ and $P_n(t)$ is the set of all polynomial of degree less than or equal to n. Show that D is a linear transformation.

   Find the matrix representation of D with respect to the bases $\{1, t, t^2, \cdots, t^n$ and $\{1, 1+t, t+t^2, \cdots, t^{n-2}+t^{n-1}\}$ of $P_n(t)$ and $P_{n-1}(t)$ respectively.

(b) If the matrix representation of a linear transformation $T : P_3(t) \to P_3(t)$ with respect to the bases $\{1-t, t-t^2, t^2-t^3, t^3\}$ and $\{1, 1+t, t+t^2, t^2+t^3\}$ is

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{pmatrix},$$

then determine T.

(c) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by

$$T(x, y, z) = (x + 2y,\ x + y + z,\ z)$$

and let $B_1 = \{(1,1,1), (1,2,3), (2,-1,1)\}$ and $B_2 = \{(1,1,0), (0,1,1), (1,0,1)\}$ be bases for $\mathbb{R}^3$. Find the matrix representation of T with respect to the basis B_2 by using the transition matrix.

4. Define what is meant by *rank* of a matrix.

(a) Let A be an $m \times n$ matrix. Prove the following:

(i) row rank of A is equal to column rank of A;

(ii) if B is a matrix obtained by performing an elementary row operation on A, then A and B have the same rank.

(b) Find the rank of the matrix

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 2 & a+1 & 3 & a-1 \\ -3 & a-2 & a-5 & a+1 \\ a+2 & 2 & a+4 & -2a \end{pmatrix}$$

for each possible value of the scalar a.

(c) Find the row reduced echelon form of

$$\begin{pmatrix} 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 3 & 3 & 0 & 2 \\ 2 & 1 & 3 & 3 & -1 & 3 \\ 2 & 1 & 1 & 1 & -2 & 4 \end{pmatrix}.$$

5. (a) Define the following terms as applied to an $n \times n$ matrix $A = (a_{ij})$.

   (i) cofactor A_{ij} of an element a_{ij};

   (ii) adjoint of A.

   Prove with the usual notations that

$$A \cdot (adj A) = (adj A) \cdot A = det A \cdot I,$$

   where I is the $n \times n$ identity matrix.

   (b) Prove that if B is a matrix obtained from a square matrix A by

   (i) multiplying a row of A by a scalar $\alpha (\neq 0)$ then $det B = \alpha \; det A$.

   (ii) interchanging two rows of A, then $det B = -\; det A$.

   (c) Let A be an n-square matrix with all elements equal to a. Prove that

   i. $det(A + \lambda I) = \lambda^{n-1}(na + \lambda)$;

   ii. $(A+\lambda I)^{-1} = \dfrac{1}{\lambda(na+\lambda)} \begin{bmatrix} (n-1)a+\lambda & -a & \cdots & -a \\ -a & (n-1)a+\lambda & \cdots & -a \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ -a & -a & \cdots & (n-1)a+\lambda \end{bmatrix}$

6. (a) State the necessary and sufficient condition for a system of linear equations to be consistent.

   Reduce the augmented matrix of the following system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

   to its row reduced echelon form and hence determine the conditions on non-zero scalars $a_{11}, a_{12}, a_{21}, a_{22}, b_1$ and b_2 such that the system has

   (i) a unique solution;

   (ii) no solution;

   (iii) more than one solution.

(b) Show that the system of equations

$$\begin{aligned} x_1 - 3x_2 + x_3 + cx_4 &= b \\ x_1 - 2x_2 + (c-1)x_3 - x_4 &= 2 \\ 2x_1 - 5x_2 + (2-c)x_3 + (c-1)x_4 &= 3b+4 \end{aligned}$$

is consistent, for all values of b if $c \neq 1$. Find the value of b for which the system is consistent if $c = 1$ and obtain the general solution for these values.

(c) State and prove Crammer's rule for 3×3 matrix and use it to solve

$$\begin{aligned} 3x_1 + x_2 + x_3 &= 3 \\ 3x_1 + 2x_2 + 2x_3 &= 5 \\ 2x_1 - 3x_2 - 2x_3 &= 1. \end{aligned}$$