EASTERN UNIVERSITY, SRI LANKA

FIRST EXAMINATION IN SCIENCE – 2002/03 & 2002/03(A

FIRST SEMESTER (June/July,2003)

MT 101 - FOUNDATION OF MATHEMATICS

Answer all questions **Time : Three hours**

Q1.

(*a*) Using the laws of algebra of propositions in logic, show that:

(*i*) $> (p \vee q) \vee (> p \wedge q) \equiv (> p)$;

(*ii*) $p \vee (p \wedge q) \equiv p$. **[10 marks each]**

(*b*) Determine the truth value of each of the following statements where $A = \{n : 1 \leq n \leq 10,\ n \text{ is a positive integer}\}$. Justify your answers.

(*i*) $(\forall\, x \in A)\ (\exists\, y \in A)(x + y < 14)$;

(*ii*) $(\forall\, x \in A)\ (\forall\, y \in A)(x + y < 14)$;

(*iii*) $(\forall\, y \in A)(x + y < 14)$. **[5 marks each]**

(*c*) Negate each of the following statements:

(*i*) If the teacher is absent, then some students do not complete their homework.

(*ii*) All the students completed their homework and the teacher is present.

(*iii*) Some of the students did not complete their homework or the teacher is absent.

[10 marks each]

(*d*) Test the validity of the following argument:

If I study, then I will not fail mathematics.
If I do not play basketball, then I will study.
But I failed mathematics.
....
Therefore I must have played basketball. **[35 marks]**

Q2.

Define the following:

- The difference, $A \backslash B$, of two sets A and B.
- Symmetric difference, $A \Delta B$, of two sets A and B.
- Power set, $\mathrm{P}(A)$, of a set A. **[5 marks each]**

Prove the following:

(*i*) $A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$;

(*ii*) $A \Delta B = (A \cup B) \backslash (A \cap B)$;

(*iii*) $(A\Delta B)\cap(A\cap B)=\phi$;

(*iv*) If $A\Delta B = A\Delta C$ then $B = C$;

(*v*) $\mathsf{P}(A)\cap\mathsf{P}(B) = \mathsf{P}(A\cap B)$. **[15 marks each]**

Construct suitable examples to show that $A\cup(B\backslash C)\neq(A\cup B)\backslash(A\cup C)$ and

$\mathsf{P}(A)\cup\mathsf{P}(B)\neq\mathsf{P}(A\cup B)$. **[10 marks]**

Q3.

Define the following:

- An equivalence relation on a set,
- An equivalence class of an element in a set. **[5 marks each]**

(*a*) Suppose Ω is a collection of relations R on a set A and let T be the intersection of the relations R, that is $T=\cap\{R : R\in\Omega\}$. Prove that

(*i*) if every R is symmetric, then T is symmetric.

(*ii*) if every R is transitive, then T is transitive. **[15 marks each]**

(*b*) A relation R on a set of rational numbers, Q, is defined by xRy if and only if $x^2y-xy^2=x^2-y^2$. Show that R is an equivalence relation on Q. **[25 marks]**

(*c*) Let $A=\{1,2,3,\ldots 14,15\}$. Let $\approx$ be the equivalence relation on $A\ X\ B$ defined by $(a,b)\approx(c,d)$ if and only if $ad=bc$. Find the equivalence class of $(3,2)$. **[25 marks]**

(*d*) If a relation R on a set A is reflexive then show that $R\cap R^{-1}$ is not empty. **[10 marks]**

Q4.

Define the terms *injective*, *surjective*, and *bijective* as applied to a mapping. **[15 marks]**

(*a*) Let $f: S\rightarrow T$ be a mapping. Prove that

(*i*) f is injective if and only if $f(A)\cap f(S\backslash A)=\phi,\ \forall\ A\subseteq S$; **[15 marks]**

(*ii*) if f is injective then $f(A\cap B)=f(A)\cap f(B),\ \forall\ A,B\subseteq S$. **[10 marks]**

(*b*) Let $f: A\rightarrow B$ and $g: B\rightarrow C$ be two mappings. Prove the following:

(*i*) If $g\circ f$ is one-to-one, then f is one-to-one;

(*ii*) If $g\circ f$ is onto, then g is onto. **[15 marks each]**

(*c*) Prove that the mapping $f:\mathsf{R}\rightarrow\mathsf{R}$, defined by

$$f(x)=\left\{\begin{array}{ll} x^4 & \text{if } x\geq 0, \\ x(2-x) & \text{if } x\leq 0, \end{array}\right\}$$

is bijective and hence find its inverse **[20 marks]**

(*d*) Suppose $f: A\rightarrow B$ is a constant function. When will f be: (*i*) one-to-one, (*ii*) onto?
[10 marks]

Q5.

Define the following:

- A *partial order* on a set.
- *Supremum* of a set.
- *Infinimum* of a set. **[5 marks each]**

When does a partially ordered set become a totally ordered set? **[5 marks]**

(*a*) Consider the relation R on the set of integers Z, which is defined in such a way that aRb if and only if $b = a^r$ for some positive integer r. Show that R is a partial order on $\mathbb{Z}$. **[30 marks]**

(*b*) Show that if R defines a partial order on a set A then R^{-1} also defines a partial order on A. **[20 marks]**

(*c*) If A and B are two totally ordered sets then prove that $A \times B$ is also a totally ordered set. **[10 marks]**

(*d*) Let $A = \{2,3,4,6,8,16,32,64\}$ and a relation R on A be defined by $xRy \Leftrightarrow x$ divides y. Find the supremum and infinimum (if exists) for a subset $B = \{2,4,8\}$ of A. **[20 marks]**

Q6.

Define the following:

- *Greatest common divisor*, gcd, *of two integers a and b,*
- *The greatest integer of a real number x,*
- *The least common multiple*,lcm, *of two integers a and b.* **[5 marks each]**

(*a*) If $a, b,$ and c are integers and c/ab then prove that c/b, where a and c are relatively primes. Hence show that if a and b are integers, p/ab, and p/a then p/b, where p is a prime. **[20 marks]**

(You may use the result that if $d = \gcd(a,b)$ then there exist integers x and y such that $ax + by = d$)

(*b*) If p and q are primes and p/q then prove that $p = q$. Hence, or otherwise, show that if $p/(q_1 q_2 \cdots q_r)$, where p and the q_i's are primes, then p is equal to one of the q_i's. **[20 marks]**

(*c*) Explain whether it is possible to have 100 coins made up of c cents, d dimes, and q quarters, be worth exactly \$5. (Here 1 dime = 10 cents, 1 quarter = 25 cents) **[15 marks]**

(*d*) State the result involving gcd and lcm of two integers and use it to find the lcm of $2m + 1$ and $2m - 1$, where $m \in \mathbb{Z}$. **[20 marks]**

(*e*) What values of x will satify the linear congruence $2x \equiv 1 \pmod 7$? **[10 marks]**