EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST EXAMINATION IN SCIENCE - 2008/2009

SECOND SEMESTER (March./April,2011)

EXTMT 102 - ANALYSIS I (Real Analysis)

EXTERNAL DEGREE

Answer all questions — Time: Three hours

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1. (a) i. Define the terms *Supremum* and *Infimum* of a non-empty subset of $\mathbb{R}$.

   ii. State the *Completeness* property of $\mathbb{R}$.

   Prove that every non-empty bounded below subset of $\mathbb{R}$ has an infimum.

   (b) i. Prove that an upper bound u of a non-empty bounded above subset S of $\mathbb{R}$ is the supremum of S if and only if for every $\epsilon > 0$, there exists an $x \in S$ such that $x > u - \epsilon$.

   ii. Let a subset S_k of $\mathbb{R}$, $k \in \mathbb{N}$ be defined by

$$S_k = \{t : t \in \mathbb{N},\ 1 \leq t \leq k\} \cup \{k + r : r \in \mathbb{N}\}.$$

   Prove that $S_k = \mathbb{N}$.

   (c) Find the supremum and infimum of the set

$$\left\{ \frac{2}{17}\left(1 - \frac{1}{13^n}\right) : n \in \mathbb{N} \right\},$$

if they exist.

2. State and prove the *Archimedean property.* Hence prove the following:

   (a) If $x \in \mathbb{R}$, then there exists a unique element $p \in \mathbb{Z}$ such that $p - 1 \leq x < p$.

   (b) There exists an irrational number $x \in \mathbb{R}$ such that $x^2 = 2$.

   (c) If $x, y \in \mathbb{R}$ with $x < y$, then there exists a rational number p such that $x < p < y$ and hence $x < q < y$ for some irrational number q.

3. (a) Define the terms *monotone sequence* and *Cauchy sequence.*

   (b) Let a sequence (x_n) be defined inductively by

   $$x_1 = 4, x_{n+1} = \frac{1}{10}\left(x_n{}^2 + 21\right), \forall\, n \in \mathbb{N}.$$

   Show that

      i. $3 < x_n < 7, \forall\, n \in \mathbb{N}$;

      ii. (x_n) is decreasing.

   Deduce that (x_n) converges and find its limit.

   (c) Prove that a sequence of real numbers is Cauchy if and only if it is convergent.

   Show that the sequence (x_n) given by $x_n = \left(n + \frac{(-1)^n}{n}\right)$ is not a Cauchy sequence.

4. (a) Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function. Define what is meant by $f(x) \to l$ as $x \to x_0,\ x_0 \in A$.

   Prove that $\lim\limits_{x \to 2} \left(2x^2 - x + 1\right) = 7$.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $\lim\limits_{x \to a} f(x) = l(\neq 0)$.

Prove the following:

i. there exists $\delta > 0$ such that $\frac{|l|}{2} < |f(x)| < \frac{3|l|}{2}$, for all x such that $0 < |x - a| < \delta$;

ii. $\lim_{x \to a} \frac{1}{f(x)} = \frac{1}{l}$, if $f(x) \neq 0, \forall\, x \in \mathbb{R}$.

5. (a) Explain in terms of ϵ, δ what is meant to say that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point $x_0 \in \mathbb{R}$.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin x, \forall\, x \in \mathbb{R}$. Prove that f is continuous on $\mathbb{R}$.

(b) Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$. Prove that it is bounded on $[a, b]$.

Is the converse result true? Justify your answer.

(c) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ -x & \text{if } x \in \mathbb{Q}^c. \end{cases}$$

is not continuous in $\mathbb{R}$ except at the point $x = 0$.

(**Hint**: Let $a \in A(\subseteq \mathbb{R})$ and let $f : A \to \mathbb{R}$, then f is not continuous at a if and only if there exist a sequence (x_n) in A that converges to a but the sequence $(f(x_n))$ does not converges to $f(a)$.)

6. (a) Suppose that f and g are continuous on $[a, b]$ differentiable on (a, b) and $g'(x) \neq 0$, for all $x \in (a, b)$. Prove that there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Deduce that

$$\lim_{x \to d} \frac{f(x)}{g(x)} = \lim_{x \to d} \frac{f'(x)}{g'(x)}, \text{ if } f(d) = g(d) = 0 \text{ for some } d \in (a, b).$$