EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

EXTERNAL DEGREE EXAMINATION IN SCIENCE - 2008/2009

FIRST YEAR SECOND SEMESTER (Jan./Mar., 2011)

EXTMT 104 - DIFFERENTIAL EQUATIONS AND FOURIER SERIES

Answer all questions　　　　Time : Three hours

1. (a) State the necessary and sufficient condition for the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

to be exact.

Hence solve the following equation

$$\left(3x^4y^2 - \frac{1}{y}\right)\frac{dy}{dx} + 4x^3y^3 + \frac{1}{x} = 0.$$

(b) Find the general solution of the following ordinary differential equations:

i. $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0;$

ii. $x\,\frac{dy}{dx} + y = y^2 \log x.$

2. (a) If $F(D) = \sum_{i=0}^{n} p_i D^i$, where $D = \frac{d}{dx}$ and p_i, $i = 1, 2, ..., n$, are constants with $p_0 \neq 0$. Prove the following formulas:

i. $\frac{1}{F(D)} e^{\alpha x} = \frac{1}{F(\alpha)} e^{\alpha x}$, where α is a constant and $F(\alpha) \neq 0$;

ii. $\frac{1}{F(D)} e^{\alpha x} V = e^{\alpha x} \frac{1}{F(D+\alpha)} e^{\alpha x} V$, where V is a function of x.

(b) Find the general solution of the following ordinary differential equations by using the results in (a).

i. $(D^3 - 3D - 2)y = 540x^3 e^{-x}$;

ii. $(D^3 + 4D^2 + 4D)y = 8e^{-2x}$.

3. (a) Let $x = e^t$. Show that

$$x\, D\, y \equiv \mathcal{D} y,$$

and

$$x^2 D^2 y \equiv (\mathcal{D}^2 - \mathcal{D})y.$$

where $\mathcal{D} \equiv \frac{d}{dt}$.

Use the above results to find the general solution of the following ordinary differential equation

$$(x^2 D^2 - 3xD + 4)y = x + x^2 \ln x.$$

(b) Solve the following simultaneous ordinary differential equations:

$$(5D + 4)y - (2D + 1)z = e^{-x},$$
$$(D + 8)y - 3z = 5e^{-x}.$$

4. Use the method of Frobenius to obtain two linearly independent solutions in series for the following ordinary differential equation

$$4x \frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 7y = 0.$$

5. (a) Write down the condition of integrability of the total differential equation

$$P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0.$$

Solve the differential equation

$$y^2\, dx - z\, dy + y\, dz = 0.$$

(b) Solve the following differential equations:

i. $\dfrac{dx}{z(x+y)} = \dfrac{dy}{z(x-y)} = \dfrac{dz}{x^2+y^2}$;

ii. $\dfrac{dx}{xy} = \dfrac{dy}{y^2} = \dfrac{dz}{zxy - 2x^2}$.

(c) Apply Charpit's method or otherwise to find the complete and the singular solution of the non-linear partial differential equation

$$2z + p^2 + qy + 2y^2 = 0.$$

(d) Solve the non-linear partial differential equation

$$pz - q^2 = 1.$$

You may assume that $z = F(x + ay) = F(u)$, where $u = x + ay$ and a is arbitrary constant.

6. (a) Obtain the Fourier series expansion of

$$f(x) = \begin{cases} 2x & \text{when} \quad 0 \leqslant x < 3, \\ 0 & \text{when} \quad -3 < x < 0. \end{cases}$$

(b) Find the finite Fourier sine transform and the finite Fourier cosine transform of $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial^2 u}{\partial x^2}$, where u is a function of x and t for $0 < x < l, \quad t > 0$.

(c) Use part (b) to show that the solution of the partial differential equation

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2},$$

subject to the boundary condition:

$V(0,t) = 0, \quad V(4,t) = 0, \quad V(x,0) = 2x$, where $0 < x < 4, \ t > 0$, is

$$V(x,t) = \frac{-16}{\pi} \sum_{n=1}^{\infty} \frac{1}{n}\, e^{\dfrac{-n^2\pi^2 t}{16}} \cos n\pi \sin \frac{n\pi x}{4}.$$