EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

EXTERNAL DEGREE EXAMINATION IN SCIENCE - 2008/2009

THIRD YEAR SECOND SEMESTER (Jan./ Apr., 2010)

EXT MT 301 - GROUP THEORY

Answer all questions

Time : Three hours

1. (a) Define the term *group*.

(b) i. Let H be a non-empty subset of a group G. Prove that, H is a subgroup of G if and only if $ab^{-1} \in H, \ \forall\, a, b \in H$.

ii. Let H and K be two subgroups of a group G. Prove that HK is a subgroup of G if and only if $HK = KH$.

iii. Let H and K be two subgroups of a group G. Is $H \cup K$ a subgroup of G? Justify your answer.

iv. Let $\{H_\alpha\}_{\alpha \in I}$ be arbitrary family of subgroups of a group G. Prove that $\bigcap_{\alpha \in I} H_\alpha$ is a subgroup of G.

2. State and prove the *Lagrange's theorem* for a finite group G.

(a) If every non-identity element of a group G has order 2, show that G is abelian.

(b) Let x and y be elements of a group G. Show that the element $x^{-1}yx$ has the same order as y.

(c) Let x and y be elements of a group, with the order of x is 5. Show that if x^3 and y commute then x and y commute.

(d) Let G be a non-abelian group of order 10. Prove that G contains at least one element of order 5.

3. State the *first isomorphism theorem.*

Let H and K be two normal subgroups of a group G such that $K \subseteq H$. Prove that:

(a) $K \trianglelefteq H$;

(b) $H/K \trianglelefteq G/K$;

(c) $\frac{G/K}{H/K} \cong G/H$.

4. (a) Let G be a group and $g_1,\ g_2 \in G$. Define a relation " $\sim$ " on G by

$$g_1 \sim g_2 \Leftrightarrow \exists g \in G \text{ such that } g_2 = g^{-1}\, g_1\, g.$$

Prove that " $\sim$ " is an equivalence relation on G.

Given $a \in G$, let $\Gamma(a)$ be denote the equivalence class of a. Show that:

i. $|\Gamma(a)| = |G : C(a)|$, where $C(a) = \{x \in G |\ ax = xa\}$;

ii. $a \in Z(G) \quad \Leftrightarrow \quad \Gamma(a) = \{a\}$, where $Z(G)$ is the center of the group G.

(b) Write down the class equation of a finite group G. Hence or otherwise, prove that the center of G is non-trival if the order of G is p^n, where p is a positive prime number.

5. (a) Define the term *p-group*.

Let G be a finite abelian group and let p be a prime number which divides the order of G. Prove that G has an element of order p.

(b) Let G' be the commutator subgroup of a group G. Prove the following:

i. G is abelian if and only if $G' = \{e\}$, where e is the identity element of G.

ii. G' is a normal subgroup of G.

iii. G/G' is abelian.

6. (a) Define the term *permutation* as applied to a group.

i. Prove that the permutation group on n symbols, S_n, is a finite group of order $n!$.

Is S_n abelian for $n > 2$? Justify your answer.

ii. Prove that the set of even permutations A_n forms a normal subgroup of S_n. Hence show that $\frac{S_n}{A_n}$ is a cyclic group of order 2.

iii. Express the permutation σ in S_8 as a product of disjoint cycles, where

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 7 & 4 & 2 & 8 & 1 & 6 \end{pmatrix}.$$