EASTERN UNIVERSITY, SRI LANKA
DEPARTMENT OF MATHEMATICS
EXTERNAL DEGREE EXAMINATION IN SCIENCE -2008/2009
FIRST YEAR, FIRST SEMESTER (July/August, 2010)
EXTMT 103 - VECTOR ALGEBRA & CLASSICAL MECHANICS I

Answer all Questions

Time: Three hours

1. (a) Define the following terms:

   i. collinear vectors;

   ii. coplanar vectors.

   (b) Let $\underline{a}, \underline{b}, \underline{c}$ be three vectors such that $\underline{a}$ is perpendicular to both $\underline{b}$ and $\underline{c}$, and $|\underline{b}| = |\underline{c}|$. Show that the equation of the plane through the three points whose position vectors are $\underline{a}, \underline{b}$ and $\underline{c}$, is

$$\left\{\frac{\underline{a}}{|\underline{a}|^2} + \frac{\underline{b}+\underline{c}}{|\underline{b}||\underline{c}| + \underline{b}\cdot\underline{c}}\right\}\cdot\underline{r} = 1,$$

   where $\underline{r}$ is the position vector of any point on the plane.

   Hence, find the equation of the plane through the points $(1, 1, 1,), (-1, 2, -1), (-1, -1, 2)$.

2. (a) If $\underline{A}$ and $\underline{B}$ are differentiable functions of a scalar u, prove:

   i. $\frac{d}{du}(\underline{A}\cdot\underline{B}) = \underline{A}\cdot\frac{d\underline{B}}{du} + \frac{d\underline{A}}{du}\cdot\underline{B}$;

   ii. $\frac{d}{du}(\underline{A}\wedge\underline{B}) = \underline{A}\wedge\frac{d\underline{B}}{du} + \frac{d\underline{A}}{du}\wedge\underline{B}$.

   Hence prove that

$$\frac{d}{du}(\underline{A}\cdot\underline{B}\wedge\underline{C}) = \underline{A}\cdot\underline{B}\wedge\frac{d\underline{C}}{du} + \underline{A}\cdot\frac{d\underline{B}}{du}\wedge\underline{C} + \frac{d\underline{A}}{du}\cdot\underline{B}\wedge\underline{C}.$$

   (b) If $\underline{r}\wedge d\underline{r} = \underline{0}$, then prove that $\hat{\underline{r}} =$ constant.

(c) Find the radius of curvature(ρ) and the torsion(τ) for the space curve $x = t,\ y = t^2,\ z = \frac{2}{3}t^3$.

3. (a) Define the following terms:

   i. the divergence of the vector field $\underline{F}$;

   ii. the curl of the vector field $\underline{F}$.

   (b) Prove the following identities:

   i. $\underline{\nabla} \cdot (\underline{A} + \underline{B}) = \underline{\nabla} \cdot \underline{A} + \underline{\nabla} \cdot \underline{B}$;

   ii. $\underline{\nabla} \cdot (\phi \underline{A}) = (\underline{\nabla}\phi) \cdot \underline{A} + \phi(\underline{\nabla} \cdot \underline{A})$,

   where $\underline{A} = \underline{A}(x, y, z)$, $\underline{B} = \underline{B}(x, y, z)$ are the vector functions and $\phi = \phi(x, y, z)$ is a scalar function.

   (c) Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$, $|\underline{r}| = r$ and $\underline{c}$ be a constant vector. Evaluate the followings:

   i. $\underline{\nabla}(\underline{c} \cdot \underline{r})$;

   ii. $curl(\underline{c} \wedge \underline{r})$.

   Hence show that

$$curl\left(\frac{\underline{c} \wedge \underline{r}}{r^3}\right) = 3\underline{r}\,\frac{(\underline{c} \cdot \underline{r})}{r^5} - \frac{\underline{c}}{r^3}.$$

4. State the *Stoke's theorem* and the *divergence theorem.*

   (a) If S is any open surface bounded by a simple closed curve C and $\underline{B}$ is any vector then prove that

$$\oint_C d\underline{r} \wedge \underline{B} = \int\int_S (\underline{n} \wedge \underline{\nabla}) \wedge \underline{B}\ ds.$$

   (b) Using the *Stoke's theorem*, evaluate $\int\int_S (\underline{\nabla} \wedge \underline{A}) \cdot \underline{n}\ ds$, where $\underline{A} = (x - z)\underline{i} + (x^3 + yz)\underline{j} - 3xy^2\underline{k}$ and S is the surface of the cone $z = 2 - \sqrt{x^2 + y^2}$ above the xy plane.

   (c) Evaluate $\int\int_S [(x^3 - yz)\underline{i} - 2x^2y\underline{j} + 2\underline{k}] \cdot \underline{n}\ ds$ by using divergence theorem, where S denotes the surface of a cube bounded by the coordinate planes and the planes $x = y = z = a$.

5. Obtain the radial and transverse components of the velocity and acceleration of a particle in the polar co-ordinate system.

A light inextensible string AOB passes through a smooth ring at a point O, on a smooth horizontal table and two particles, m_1 and m_2 are attached to it's ends A and B. Initializing the particles lie on the table with $OA(=x)$, and $OB(=y)$ and AOB a straight line. The mass m_1 is now projected horizontally with velocity v perpendicular to OA. If the string remains in contact with the ring, and all the motion takes place in a horizontal plane, prove that the mass m_2 reaches the ring with velocity

$$\frac{v}{x+y}\sqrt{\frac{m_1 y(2x+y)}{(m_1+m_2)}}.$$

6. A particle moves in a plane with the velocity v and the tangent to the path of the particle makes an angle ψ with a fixed line in the plane. State the velocity and acceleration components of the particle in intrinsic coordinate. By using these, show that the components of acceleration along the tangent and perpendicular to it are given by $v\dfrac{dv}{ds}$ and $v^2\dfrac{d\psi}{ds}$ respectively.

A particle slides on a rough wire in the form of the cycloid $s = 4a\sin\psi$ which is fixed in a vertical plane with axis vertical and vertex downwards. It is projected from the vertex with speed u so that it comes to rest at the cusp $(\psi = \pi/2)$. Show that

$$e^{\mu\pi} = \mu^2 + (\mu^2+1)u^2/4ag,$$

where μ is the co-efficient of the friction.