EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

EXTERNAL DEGREE EXAMINATION IN SCIENCE - 2008/2009

SECOND YEAR SECOND SEMESTER (Jan./Apr., 2010)

EXTMT 217 - MATHEMATICAL MODELING

Answer all questions   Time: Two hours

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1. Write down the steps involved in a mathematical model building process.

   Chemicals A and B are engaged in a reaction. To describe this reaction, formulate a mathematical model as a second-order ordinary differential equation.

   A compound C is formed when two chemicals A and B combined. The resulting reaction between the two chemicals is such that for each gram of A, 4 grams of B is used. It is observed that 30 grams of the compound C is formed in 10 minutes. Determine the amount of C at any time if the rate of the reaction is proportional to the amounts of A and B remaining, and initially there are 50 grams of A and 32 grams of B. How much of the compound C is present at 15 minutes ? Interpret the solution as $t \longrightarrow \infty$.

2. Pollution in our lakes and rivers has become a major problem particularly over the last 50 years. In order to improve this situation, briefly describe the general compartment model with a single compartment, the lake.

A large tank contains 100 litres of salt water. Initially S_0 Kg of salt is dissolved. Salt water flows in to the tank at the rate of 10 litres per minute, and the concentration $C_{\text{in}(t)}$ (Kg of salt/ litre) of this incoming water-salt mixture varies with time. Assume that the solution in the tank is thoroughly mixed and that the salt solution flows out at the same rate at which it flows in, that is, the volume of water-salt mixture in the tank remains constant.

(a) Find a differential equation for the amount of salt in the tank at any time t.

(b) Using the technique of integrating factors, solve this initial value problem on the interval $[0, t]$, in terms of an arbitrary $C_{\text{in}(t)}$.

(c) Let $C_{\text{in}(t)}$ be a sinusoidal function, say $C_{\text{in}(t)} = 0.2 + 0.1 \sin t$, evaluate the above integral.

3. Suppose a x - force and y - force are engaged in combat. Let $x(t)$ and $y(t)$ denote the respective strength of the forces at time t, when t is measured in days from the start of the combat. Conventional combat model is given by

$$\frac{d}{dt}x(t) = -ax(t) - by(t) + P(t),$$

$$\frac{d}{dt}y(t) = -dy(t) - cx(t) + Q(t),$$

where a, b, c and d are arbitrary constants. Explain the terms involved in these equations. By assuming that there is no reinforcement arrived and no operational losses occur, obtain a simplified model and hence show that

$x(t) = x_0 \cosh(\beta t) - \gamma y_0 \sinh(\beta t)$, where $\beta = \sqrt{bc}, \gamma = \sqrt{b/c}$ and x_0, y_0 are the initial strength of the respective forces.

4. Distinguish between exponential and logistic population growth. Give the equations for each.

The fish population in a certain part of the sea can be separated into prey population (food fish) $x(t)$ and predator population (Selachians) $y(t)$. The model governing the interaction of the selachians and food fish in the absence of fishing is given by

$$\left.\begin{aligned} \frac{dx}{dt} &= ax - bxy \\ \frac{dy}{dt} &= -cy + dxy \end{aligned}\right\} \quad \text{(i)}$$

where a, b, c and d are arbitrary constants.

(a) Explain the terms involved in this model.

(b) Show that $\frac{y^a}{e^{by}} \cdot \frac{x^c}{e^{dx}} = k$, where k is a constant.

(c) Let $x(t)$ and $y(t)$ be the periodic solution of (i) .

If $\overline{x} = \frac{1}{T}\int_0^T x(t)dt$ and $\overline{y} = \frac{1}{T}\int_0^T y(t)dt$, then show that $\overline{x} = \frac{c}{d}$ and $\overline{y} = \frac{a}{b}$, where T is the period.

Hence show that a moderate amount of fishing increase the average number of food fish and decrease the average number of selachians.