EASTERN UNIVERSITY, SRI LANKA

EXTERNAL DEGREE EXAMINATION IN SCIENCE

FIRST YEAR FIRST SEMESTER -2004/2005

(May/ Jun., 2008)

MT 103 - VECTOR ALGEBRA & CLASSICAL MECHANICS I

(Proper and Repeat)

Answer all questions **Time : Three hours**

1. (a) Define the terms 'Scalar product' and 'Vector product' of two vectors.

For any three vectors $\underline{a}, \underline{b}, \underline{c}$, prove that the identity

$$\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \cdot \underline{c})\,\underline{b} - (\underline{a} \cdot \underline{b})\,\underline{c}.$$

(b) $\underline{p}$, $\underline{q}$ and $\underline{r}$ are three non-null vectors such that $\underline{r} - (\underline{p} \wedge \underline{q}) = \alpha \underline{q}$ and $\underline{p} \cdot \underline{q} = 0$, where α is a scalar. Show that

$$\underline{p} = \underline{q} \wedge \frac{\underline{r}}{q^2} \quad \text{and} \quad \alpha = \frac{\underline{q} \cdot \underline{r}}{q^2}.$$

(c) If a vector $\underline{r}$ is resolved into components parallel and perpendicular to a given vector $\underline{a}$, show that the decomposition is

$$\underline{r} = \frac{(\underline{a} \cdot \underline{r})\underline{a}}{a^2} + \frac{\underline{a} \wedge (\underline{r} \wedge \underline{a})}{a^2}.$$

2. (a) Define the following terms;

   i. the **gradient** of a scalar field ϕ,

   ii. the **divergence** of a vector field $\underline{F}$,

   iii. the **curl** of a vector field $\underline{F}$.

(b) Prove that

   i. div $(\phi\, \underline{F}) = \text{grad}\, \phi \cdot \underline{F} + \phi\, \text{div}\, \underline{F}$,

   ii. curl $(\phi\, \underline{F}) = \phi\, \text{curl}\, \underline{F} + \text{grad}\, \phi \wedge \underline{F}$.

(c) Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ and $r = |\underline{r}|$ and let $\underline{a}$ be a constant vector. Evaluate t following:

   i. $\text{grad}(\underline{a} \cdot \underline{r})$;

   ii. $\text{curl}(\underline{a} \wedge \underline{r})$.

Hence show that

   i. $\text{grad}\left(\dfrac{\underline{a} \cdot \underline{r}}{r^3}\right) = \dfrac{\underline{a}}{r^3} - \dfrac{3(\underline{a} \cdot \underline{r})}{r^5}\, \underline{r}$,

   ii. $\text{curl}\left(\dfrac{\underline{a} \wedge \underline{r}}{r^3}\right) = \dfrac{2\underline{a}}{r^3} + \dfrac{3\, \underline{a} \wedge \underline{r}}{r^5} \wedge \underline{r}$.

3. (a) State the **Divergence theorem** and use it to evaluate $\displaystyle\int\!\!\int_S \underline{A} \cdot \underline{n}\; dS$, whe $A = 2x^2y\underline{i} - y^2\underline{j} + 4xz^2\underline{k}$ and S is the surface of the cylinder $x^2 + y^2 = 9$ includ in the first octant between $x = 0$ and $x = 2$.

(b) State the **Green's Theorem**.

Verify the Green's theorem in plane for

$$\int_C [(x^2 - xy^3)\, dx + (y^2 - 2xy)dy]$$

where C is in the square with vertices $(0,0)$, $(2,0)$, $(2,2)$, $(0,2)$.

4. Prove that the radial and transverse component of the acceleration of a particle in a plane in terms of polar co-ordinates (r, θ) are

$$\ddot{r} - r\dot{\theta}^2 \text{ and } \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) \text{ respectively.}$$

(a) The velocities of a particle along and perpendicular to a radius vector from a fixed origin are λr^2 and $\mu\theta^2$. Find the polar equation of the path of the particle and also the components of acceleration in terms of r and θ.

(b) A light inextensible string of length $2a$ passes through a smooth ring at a point O, on a smooth horizontal table and two particles, each of mass m, attached to it's ends A and B. Initially the particles lie on the table with $OA = OB = a$ and AOB a straight line, the particle A is given a velocity u in a direction perpendicular to OA. Prove that, if r and θ are the polar co-ordinates of A at a time t with respect to the origin, then

i. $2\dfrac{d^2r}{dt^2} - \dfrac{a^2u^2}{r^3} = 0,$

ii. $2r\dfrac{dr}{dt} = u\sqrt{2(r^2 - a^2)},$

iii. $r^2 = a^2 + \dfrac{1}{2}u^2t^2.$

Find the velocity of A at the instant when B reaches the origin at O.

5. A particle moves in a plane with velocity v and the tangent to the path of the particle makes an angle ψ with a fixed line in the plane. Prove that the components of the acceleration of the particle along the tangent and perpendicular to it are $\dfrac{dv}{dt}$ and $v\dfrac{d\psi}{dt}$ respectively.

A body attached to a parachute is released from an aeroplane which is moving horizontally with velocity v_0. The parachute exerts a drag opposing motion which is k times the weight of the body, where k is a constant. Neglecting the air resistance to the motion of the body, prove that if v is the velocity of the body when its path is inclined an angle ψ to the horizontal, then

$$v = \frac{v_0 \sec\psi}{(\sec\psi + \tan\psi)^k}.$$

Prove that if $k = 1$, the body cannot have a vertical component of velocity grater than $\dfrac{v_0}{2}$.

6. Establish the equation

$$F(t) = m(t)\frac{dv}{dt} + v_0\frac{dm(t)}{dt}$$

for the motion of a rocket of varying mass $m(t)$ moving in a straight line with velocity $\underline{v}$ under a force $\underline{F}(t)$, matter being emitted at a constant rate with a velocity $\underline{v_0}$ relative to the rocket.

(a) A rocket of total mass m contains fuel of mass ϵm $(0 < \epsilon < 1)$. This fuel burns at a constant rate k and the gas is ejected backward with the velocity u_0 relative to the rocket. Find the speed of the rocket when the fuel has been completely burnt.

(b) A rain drop falls from rest under gravity through a stationary cloud. The mass of the rain drop increases by absorbing small droplets from the cloud. The rate of increment is mrv, where m is the mass, v is the speed and r is a constant. Show that after the rain drop fallen a distance x, $rv^2 = g(1 - e^{-2rx})$.