EASTERN UNIVERSITY, SRI LANKA

EXTERNAL DEGREE FIRST EXAMINATION IN SCIENCE(2003/2004)

SECOND SEMESTER (October, 2007)

EXTMT 102 - REAL ANALYSIS

(Proper & Repeat)

Answer all questions **Time : Three hours**

Q1. (a) i. Define the terms "**Supremum**" and "**Infimum**" of a non-empty subset of $\mathbb{R}$.

ii. State the completeness property of $\mathbb{R}$.

(b) Prove that an upper bound u of a non-empty bounded above subset S of $\mathbb{R}$ is the supremum of S if and only if for every $\epsilon > 0$, there exists $x \in S$ such that $x > u - \epsilon$.

(c) i. Let S be a non-empty bounded above subset of $\mathbb{R}$ and let $a \in \mathbb{R}$. Define the set $a + S = \{a + x \ : \ x \in S\}$. Show that,

$$\sup(a + S) = a + \sup S.$$

ii. Find, if they exist, the Supremum and Infimum of the set $\left\{\frac{2}{17}\left(1 - \frac{1}{11^n}\right) : n \in \mathbb{N}\right\}$.

Q2. (a) State and prove the Archimedean property. Hence prove the following:

i. If $x \in \mathbb{R}$, then there exists a unique element $n \in \mathbb{Z}$ such that $n \leq x \leq n+1$.

ii. If $x, y \in \mathbb{R}$ with $x < y$, then there exists a rational number p such that $x < p < y$ and hence $x < q < y$ for some irrational number q.

(b) Define the "Inductive set".

Prove that $\mathbb{N}$ is the smallest inductive set.

Q3. (a) Define what is meant by each of the following terms as applied to a sequence of real number.

i. bounded;

ii. convergent;

iii. monotonic.

(b) Prove that, a monotone sequence $(x_n)_{n=1}^{\infty}$ of real numbers is convergent if and only if it is bounded.

(c) Let a sequence $(y_n)_{n=1}^{\infty}$ be defined inductively by

$$y_1 = 1, \ y_{n+1} = \frac{1}{4}(2y_n + 3), \ \forall \ n \in \mathbb{N}.$$

Show that

i. $y_n < 2, \forall n \in \mathbb{N}$.

ii. $(y_n)_{n=1}^{\infty}$ is strictly increasing.

Deduce that $(y_n)_{n=1}^{\infty}$ converges and find its limit.

Q4. (a) Define the term **"Cauchy sequence"** of real numbers.

(b) Show that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence if and only if it is convergent.

(c) Prove that every Cauchy sequence is bounded.

Is the converse of this result is true? Give reasons for your answer.

(d) Determine whether each of the following is a Cauchy sequence.

i. $\left(\frac{1}{n^2}\right)_{n=1}^{\infty}$;

ii. $\left(n + \frac{(-1)^n}{n}\right)_{n=1}^{\infty}$.

Q5. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Explain what is meant by the statement that the function f is continuous at 'a' ($\in \mathbb{R}$).

Prove that if f is continuous at 'a', then the function $|f|$ is also continuous at 'a'.

Is the converse of this result is true? Give reasons for your answer.

(b) Prove that a function $g : \mathbb{R} \to \mathbb{R}$ is continuous at 'a' ($\in \mathbb{R}$) if and only if for every sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}$ that converges to 'a', the sequence $(f(x_n))_{n=1}^{\infty}$ converges to $f(a)$.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q}, \\ 0 & \text{if} \quad x \in \mathbb{Q}^c, \end{cases}$$

show that f is not continuous at every point of $\mathbb{R}$.

Q6. (a) State what is meant by the statement that a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at 'a' ($\in \mathbb{R}$).

(b) State **Rolle's theorem** and use it to prove the **Mean-Value** theorem.

(c) Suppose that $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are differentiable functions on $[a, b]$ and that $f'(x) = g'(x)$ for all $x \in (a, b)$. Prove that there exists a constant k such that $f(x) = g(x) + k$ for all $x \in [a, b]$.

(d) Let a function $f : \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$ such that $f'(\alpha) = 0$ for some $\alpha \in \mathbb{R}$. Suppose that $f''(\alpha)$ exists. prove that if $f''(\alpha) > 0$, then f has a minimum at $x = \alpha$.