EASTERN UNIVERSITY, SRI LANKA

EXTERNAL DEGREE SECOND EXAMINATION

IN SCIENCE 2002/2003

Oct./Nov.' 2007

SECOND SEMESTER

EXTMT 202 - METRIC SPACE

Answer all questions **Time:Two hours**

Q1. (a) Define the following:

i. Metric Space;

ii. Complete Metric Space.

(b) Let X be a set of all bounded sequence of real numbers. Define $d : X \times X \longrightarrow \mathbb{R}$ by

$$d(x, y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i},$$

where $x = \{x_i\}_{i \in \mathbb{N}}$ and $y = \{y_i\}_{i \in \mathbb{N}}$ are two arbitrary elements of X. Show that (X, d) is a metric space.

(c) Prove that every open ball is an open set.

(d) Prove that $\mathbb{R}$ with the usual metric is complete.

Q2. (a) Let (X, d) be a metric space and let $A \subseteq X$. Define the term *Closure of A*. Prove that, the closure of A is the smallest closed set containing A.

(b) Let A, B be any two subsets of a metric space (X, d). Prove that

i. $(A^o \cap B^o) = (A \cap B)^o$.

ii. $(A^o \cup B^o) \subseteq (A \cup B)^o$.

Give an example to show $(A^o \cup B^o) \neq (A \cup B)^o$.

(c) Let (X, d) be a metric space and let $A \subseteq X$.Define the term *frontier point of A*. Prove that $Fr(A) = \overline{A} \cap \overline{A^C}$.

Q3. (a) Define the following:

i. Connected Set;

ii. Compact Set.

(b) Let (Y, d_Y) be a subspace of a metric space (X, d). Prove that Y is connected if and only if the only subset of Y both open and closed in Y is Y itself.

(c) Show that $[a, b]$ is compact in $(\mathbb{R}, |.|)$.

(d) Let $X = \mathbb{R}$ and d the usual metric. Prove that $(0, 1)$ is a compact set in $(\mathbb{R}, d)$. Justify your answer.

Q4. (a) Let (X, d_1) and (Y, d_2) be any two metric spaces and $f : X \longrightarrow Y$ be a function. Prove that f is continuous at $a \in X$ if and only if for every sequence $\{a_n\}_{n=1}^{\infty}$ in X converging to a we have $\{f(a_n)\}_{n=1}^{\infty}$ converging to $f(a)$.

(b) Let (X, d_1) and (Y, d_2) be two metric spaces. Prove that $f : X \longrightarrow Y$ is continuous if and only if $f^{-1}(G)$ is open in X whenever G is open in Y.

(c) Show that $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x \;\; \forall\, x \in \mathbb{R}$ is continuous on $\mathbb{R}$,where $(\mathbb{R}, d)$ is the usual metric space.

(d) Let (X, d_X) be a discrete metric space and let (Y, d_Y) be any metric space.. Prove that every function from X to Y is continuous on X.

(e) Prove that $f : X \longrightarrow Y$ is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)} \;\; \forall\, A \subseteq X$.