EASTERN UNIVERSITY, SRI LANKA

FIRST YEAR EXAMINATION IN SCIENCE, 2002/2003

EXTERNAL DEGREE

SECOND SEMESTER

(Sept./Oct. '2005)

EXTMT 104 - DIFFERENTIAL EQUATIONS AND FOURIER SERIES

Answer **All** Questions   Time Allowed: **3** Hours

Q1. (a) State the necessary and sufficient condition for the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

to be **exact**. [10 Marks]

Hence solve the following differential equation

$$\left(2xy\, e^{x^2 y} + y^2 e^{xy^2} + 1\right) dx + \left(x^2 e^{x^2 y} + 2xy\, e^{xy^2} - 2y\right) dy = 0.$$

[20 Marks]

(b) If $\tan x$ is a particular solution of the following non-linear Riccati differential equation

$$\frac{dy}{dx} = 1 + y^2,$$

then obtain the general solution of the differential equation .

[70 Marks]

Q2. (a) If $F(D) = \sum_{i=0}^{n} p_i D^i$, where $D \equiv \frac{d}{dx}$ and p_i, $i = 1, \ldots, n$, are constants with $p_0 \neq 0$, then prove the following formulas:

(i) $\frac{1}{F(D)} e^{\alpha x} = \frac{1}{F(\alpha)} e^{\alpha x}$, α is a constant and $F(\alpha) \neq 0$;

(ii) $\frac{1}{F(D)} e^{\alpha x} V = e^{\alpha x} \frac{1}{F(D+\alpha)} V$, where V is a function of x.

[40 Marks]

(b) Find the general solution of the following differential equations by using the results in (a).

(i) $(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 2e^x + 3e^{-x}$.

(ii) $(D^3 - 3D^2 - 6D + 8)y = xe^{-3x}$.

[60 Marks]

Q3. (a) If $x = e^t$, then show that

$$x\frac{d}{dx} \equiv \mathcal{D}, \quad x^2 \frac{d^2}{dx^2} \equiv \mathcal{D}^2 - \mathcal{D},$$

where $\mathcal{D} \equiv \frac{d}{dt}$.

[20 Marks]

Use the above results to find the general solution of the following differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12 \log x}{x^2}.$$

[30 Marks]

(b) With $D \equiv \frac{d}{dt}$, solve the following simultaneous differential equations

$$\begin{aligned} D^2x - m^2y &= 0, \\ D^2y + m^2x &= 0. \end{aligned}$$

[50 Marks]

Q4. Use the method of Frobenius to find the general solution of

$$(x-1)^2\frac{d^2y}{dx^2}+(3x^2-4x+1)\frac{dy}{dx}-2y=0$$

by expanding about $x=1$. [100 Marks]

Q5. (a) Solve the following system of differential equations:

(i) $\dfrac{dx}{x(y^2-z^2)}=\dfrac{dy}{y(z^2-x^2)}=\dfrac{dz}{z(x^2-y^2)}$;

(ii) $\dfrac{dx}{x(2y^4-z^4)}=\dfrac{dy}{y(z^4-2x^4)}=\dfrac{dz}{z(x^4-y^4)}$.

[30 Marks]

(b) Write down the condition of integrability of the total differential equation

$$P(x,y,z)\,dx+Q(x,y,z)\,dy+R(x,y,z)\,dz=0.$$

[5 Marks]

Hence solve the following equation

$$yz\log z\,dx-zx\log z\,dy+xy\,dz=0.$$

[15 Marks]

(c) Find the general solution of the following linear first-order partial differential equations:

(i) $(y-z)p+(z-x)q=y-x$;

(ii) $(x^2+y^2-yz)p-(x^2+y^2-xz)q=z(x-y)$.

[30 Marks]

(d) Apply Charpit's method or otherwise to find the complete and the singular solution of the following non-linear first-order partial differential equation

$$2xz-px^2-2qxy+pq=0.$$

Here, $p=\dfrac{\partial z}{\partial x}$ and $q=\dfrac{\partial z}{\partial y}$.

[20 Marks]

Q6. (a) Prove that if $-\pi \leq x \leq \pi$ and a is not an integer, then

$$\cos ax = \frac{2a \sin a\pi}{\pi} \left\{ \frac{1}{2a^2} - \frac{\cos x}{a^2 - 1} + \frac{\cos 2x}{a^2 - 4} - \cdots \right\}.$$

[20 Marks]

Use the above result to show that

$$\frac{a\pi}{\sin a\pi} = 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n a^2}{a^2 - n^2}.$$

[20 Marks]

(b) Use Fourier transform to solve the one-dimensional heat equation

$$\frac{\partial U}{\partial t} = 2 \frac{\partial^2 U}{\partial x^2},$$

subject to the boundary conditions

$$U(0, t) = 0, \; U(x, 0) = e^{-x}, \quad x > 0$$

and U(x,t) is bounded where $x > 0$ and $t > 0$.

[60 Marks]