EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE (1996/97)

/ EXTERNAL DEGREE

(June/August' 2004)

Re-Repeat

EXMT 303 & 304 - FUNCTIONAL ANALYSIS & TOPOLOGY

Answer five questions only selecting at least two questions from each section

Time : Three hours

Section A

1. State what is meant to say that two norms in a normed linear space are equivalent.

   (a) If $\{x_1, x_2, \cdots, x_n\}$ is a set of linearly independent vectors in a normed linear space X, then there exists a number $k > 0$ such that

   $$\| \sum_{i=1}^{n} \eta_i x_i \| \geq k \sum_{i=1}^{n} |\eta_i|$$

   for every choice of scalars $\eta_1, \eta_2, \cdots, \eta_n$.

Use this result to prove the following:

i. Every finite dimensional subspace of X is complete;

ii. Any two norms on a finite dimensional normed linear space are equivalent.

(b) In unitary space $\mathbb{C}^n$ two norms $\|.\|_1$ and $\|.\|_2$ are defined by,

$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$

$\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$

where $x = (x_1, x_2, \cdots, x_n)$.

Prove that for every $x \in \mathbb{C}^n$,

$$\frac{1}{\sqrt{n}} \leq \|x\|_1 \leq \|x\|_2 \leq \|x\|_1.$$

2. Define the term "bounded linear operator" from a normed linear space into another normed linear space.

(a) Let T be a linear operator from a normed linear space X into a normed linear space Y. Show that the following statements are equivalent.

i. T is continuous at the origin;

ii. T is continuous;

iii. T is bounded.

(b) If T is a linear operator from a normed linear space X onto a normed linear space Y, then show that the inverse operator $T^{-1} : Y \to X$ exists and is bounded if and only if there exists $k > 0$ such that

$$\| T(x) \| \geq k \| x \| \qquad \forall\, x \in X.$$

(c) Show that the operator $T : l^2 \to l^2$ defined by

$$T(x) = (\eta_i), \quad \eta_i = \frac{\zeta_i}{2^i}, \quad x = (\zeta_i)$$

is linear and bounded.

3. State the Hahn Banach theorem for normed linear spaces.

(a) Let X be a normed linear space and let $x_0 \neq 0$ be any element of X. Prove that there exists a bounded linear functional g on X such that $\| g \| = 1$ and $g(x_0) = \| x_0 \|$.

Deduce that if $f(x) = f(y)$ for every bounded linear functional on X then $x = y$.

(b) Let Y be a closed linear subspace of a normed linear space X and let $x_0 \in X \setminus Y$, and let $\delta = \inf\{\| y - x_0 \| : y \in Y\}$. Show that there exists a bounded linear functional f defined on X such that $\| f \| = 1$, $f(Y) = \{0\}$ and $f(x_0) = \delta$.

4. (a) Define the following terms in a normed linear space:

i. Schauder basis;

ii. Absolutely convergent series.

(b) Let $e_i = (0, 0, \cdots, 1^{i^{th}}, 0, \cdots)$, $i \in \mathbb{N}$. Prove that the sequence $(e_i)_{i=1}^{\infty}$ is a schauder basis for the sequence space l^p if $1 \leq p < \infty$.

(c) Prove that a normed linear space is complete if and only if every absolutely convergent series is convergent.

(d) Prove that the dual space of l^1 is l^∞.

Section B

5. Define the following terms:

- Topology on a set;
- Subspace of a topological space;
- Closed subset of a topological space.

(a) Let X be a non-empty set and let (Y, τ) be a topological space. Let $f : X \to Y$ and for $A \subseteq Y$, define $f^{-1}(A) = \{x \in X : f(x) \in A\}$. Prove that $\{f^{-1}(B) : B \in \tau\}$ is a topology on X.

Is it true that any closed set in a topological subspace is closed in its topological space? Justify your answer.

(b) Let τ_1 and τ_2 be two topologies on a non-empty set X. Is $\tau_1 \cup \tau_2$ is a topology on X? Justify your answer.

6. Define the following terms in the topological space.

- Separated set;
- Connected set.

(a) Let X be a topological space. Prove the following:

i. X is disconnected if and only if there exists a non-empty proper subset of X which is both open and closed.

ii. If X is connected and f is a continuous function defined on X then $f(X)$ is connected.

iii. X is disconnected if and only if there exists a continuous function of X onto a two point space $\{0, 1\}$.

(b) Prove that a subset of the real line $\mathbb{R}$ having more than one point is connected if and only if it is an interval.

7. Let f be a function from a topological space (X, τ_X) into a topological space (Y, τ_Y).

(a) What is meant by that f is continuous at a point $x_0 \in X$?

(b) Prove that, f is continuous on X if and only if, $f^{-1}(G)$ is open in X, for every open subset G in Y.

(c) Is it true that if f is continuous on X then the image of a closed set in X is closed in Y? Justify your answer.

(d) Prove that f is continuous on X if and only if $f^{-1}(A^0) \subseteq \{f^{-1}(A)\}^0$ for every subset A of Y.

(e) If f is continuous and Z is a subspace of X, show that the restriction of f to Z is continuous.

8. Define :

- Frechet space (T_1);
- Housdorff space (T_2).

(a) Prove that every Housdorff space is a Frechet space.

Is the converse true? Justify your answer.

(b) Prove that a topological space X is a Frechet space if and only if every singleton subset of X is closed.

(c) Prove that every subspace of a Frechet space is also a Frechet space.