EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE - 1996/97(June/July'2004)

EXTERNAL DEGREE - REPEAT

EXMT 306 - PROBABILITY & STATISTICS II

Answer five questions only **Time : 3 hours**

1. Define the term unbiasedness.

(a) Consider a distribution having a normal population with mean μ and variance σ^2.

i. Show that the sample mean $\bar{X}$ is an unbiased estimator for μ.

ii. Let $S_1^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$ and $S_2^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ be estimators for σ^2. Check the unbiasedness of S_1^2 and S_2^2.

(b) Define the term "likelihood function".

i. Let $X_1, X_2, \cdots, X_n$ be n independent observations from normal distribution with unknown mean μ and an unknown variance σ^2. Find the maximum likelihood estimators for μ and σ^2.

ii. Let us assume that the sample data follow a normal distribution with unknown mean μ and unknown variance σ^2. If $n = 12, \sum_{i=1}^{12} x_i = 180$ and $\sum_{i=1}^{12} x_i^2 = 2799$. Find the maximum likelihood estimators for μ and σ^2.

2. (a) Let X be a random variable and $U = X^2$.

Show that

$$F_U(u) = F_X(\sqrt{u}) - F_X(-\sqrt{u}),$$

where $F_U(.)$ and $F_X(.)$ are probability distribution functions of U and X respectively.

Hence show that if random variable X follows the standard normal distribution, the random variable U follows the Chi-square distribution with one degree of freedom.

(b) A random variable Y_1 is the proportional amount of gasoline stocked at the beginning of a week, and a random variable Y_2 is the proportional amount of gasoline sold during the week. The joint distribution on Y_1 and Y_2 is given by

$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \leq y_2 \leq y_1 \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the probability density function of $V = Y_1 - Y_2$, the proportional amount of gasoline remaining at the end of the week.

3. (a) State and prove the Cramer-Rao inequality.

Prove, with the usual usual notation, that

$$E\left[\left\{\frac{\partial \log f_X(x;\theta)}{\partial \theta}\right\}^2\right] = -E\left[\frac{\partial^2 \log f_X(x;\theta)}{\partial \theta^2}\right]$$

(b) Given the probability density function,

$$f(x,\theta) = [\pi\{1+(x-\theta)^2\}]^{-1} \quad ; \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

Show that the Cramer- Rao lower bound of variance of an unbiased estimator of θ is $\frac{2}{n}$. Where n is the size of the random sample from this distribution.

4. Define the term **moment generating function** of a random variable X.

The probability density function of a Gamma distribution with parameters m and λ is

$$f_X(x) = \begin{cases} \dfrac{\lambda^m x^{m-1} e^{-\lambda x}}{\Gamma(m)} & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Find the moment generating function of the Gamma distribution.

Hence find the moment generating functions of the following:

(a) Chi-square distribution with degrees of freedom n,

(b) exponential distribution with parameter λ.

Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from an exponential distribution with mean $\frac{1}{\lambda}$. Show that $T = \sum_{i=1}^{n} X_i$ follows a Gamma distribution with parameters n and λ.

Hence prove that $2T\lambda \sim \chi^2_{2n}$.

5. (a) Define the term **Conditional probability**.

i. Let A and B be two events. Show that

$P(A) = P(A/B)P(B) + P(A/B^c)P(B^c)$.

ii. The national cricket team of a country has a constant probability 0.8 of winning a match played at their home country and 0.6 of winning a match played abroad. During this season, the team plays six matches in their home country and six matches abroad. calculate the mean and variance of the number of matches that will be won by the team during the season.

(b) Show that the following statements are equivalent.

i. A and B^c are independent;

ii. A and B are independent;

iii. A^c and B^c are independent;

iv. $P(A \cup B) = 1 - P(A^c)P(B^c)$.

6. (a) Let Y be a negative binomial random variable with parameters r and p and its probability mass function be given by,

$$P(Y = y) = \begin{cases} \begin{pmatrix} y-1 \\ r-1 \end{pmatrix} p^r q^{y-r} & ; \quad y = r, r+1, r+2, \cdots \\ 0 & ; \quad \text{otherwise.} \end{cases}$$

Find,

i. the expected value of Y,

ii. the variance of Y,

iii. the moment generating function of Y.

(b) The mean muscular endurance score of a random sample of 60 subjects was found to be 145 with standard deviation of 40. Construct a 95% confidence interval for the true mean. Assume the sample size to be large enough for normal approximation. What size of sample is required to estimate the mean within 5 of the true mean with a 95% confidence?

7. (a) Let the density function of the random variable Y be given by

$$f(y) = \begin{cases} \dfrac{2}{\pi(1+y^2)} & -1 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

i. Find the distribution function.

ii. Find $E(Y)$.

(b) A function that is sometimes associated with continuous nonnegative random variables in the failure rate(or hazard rate) function. This is defined by

$$r(t) = \frac{f(t)}{1 - F(t)},$$

for a density function $f(t)$ with corresponding distribution function $F(t)$. If we think of the random variable in question as the length of life of a component, $r(t)$ is proportional to the probability of failure in a small interval at time t, given that the component has survived up to time t.

i. Show that for an exponential density function, $r(t)$ is constant.

ii. Show that for a Weibull density function

$$f(y) = \frac{m}{\alpha} y^{m-1} e^{-\frac{1}{\alpha} y^m}, \quad 0 \leq y < \infty, \quad \alpha, m > 0$$

$r(t)$ is an increasing function of t for $m > 1$.

8. (a) Define type I and type II error.

For Jones political poll $n = 15$ votes are sampled. We wish to test $H_0 : p = 0.5$ against the alternative $H_1 : p = 0.3$. The test statistic is y, the number of sampled votes favoring Jones.

If we select $\{ y ; y \leq 2 \}$ as the critical region, calculate,

i. Type I error

ii. Type II error

(b) A random sample $X_1, X_2, \cdots, X_n$ is obtained from a distribution with probability density function,

$$f(x) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)} \ ; \ 0 \leq x < \infty$$

Where α and β are unknown parameters. Estimate α and β by using the method of moments.