EASTERN UNIVERSITY, SRI LANKA

EXTERNAL DEGREE FIRST EXAMINATION IN SCIENCE - 2003/2004

SECOND SEMESTER(October, 2007)

EXTMT 104 - DIFFERENTIAL EQUATIONS AND FOURIER SERIES

(PROPER AND REPEAT)

Answer all Questions **Time: Three hours**

Q1. (a) State the necessary and sufficient condition for the differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

to be exact. Hence, solve the following differential equation

$$(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0.$$

[40 marks]

(b) If a solution of the differential equation

$$4x^2y'' + 4xy' + (4x^2 - 1)y = 0$$

is

$$y_1(x) = \frac{\sin x}{\sqrt{x}}, \quad 0 < x < \pi,$$

find a second solution $y_2(x)$ using an appropriate formula. [Give the formula that you use]

[40 marks]

(c) Solve

$$(x-2)\frac{dy}{dx} = y + 2(x-2)^3.$$

[20 marks]

Q2. (a) If $F(D) = \sum_{i=0}^{n} p_i D^i$, where $D \equiv \frac{d}{dx}$ and p_i, $i = 1, 2, \ldots, n$ are constants with $p_0 \neq 0$, prove the following formulas:

(i) $\frac{1}{F(D)}\, e^{\alpha x} = \frac{1}{F(\alpha)}\, e^{\alpha x}$, where α is a constant and $F(\alpha) \neq 0$;

(ii) $\frac{1}{F(D)}\, e^{\alpha x} V = e^{\alpha x} \frac{1}{F(D+\alpha)} V$, where V is a function of x. [40 marks]

(b) Find the general solution of the following differential equations by using the results in (a).

(i) $(D^4 - 2D^2 + 1)y = 40 \cosh x$.

(ii) $(D^2 - 1) = xe^{3x}$.

[60 marks]

Q3. (a) Use the suitable substitution to show that

$$x\frac{d}{dx} \equiv \mathcal{D}, \quad x^2\frac{d^2}{dx^2} \equiv \mathcal{D}(\mathcal{D}-1) \quad \text{and} \quad x^3\frac{d^3}{dx^3} \equiv \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2),$$

where $\mathcal{D} \equiv \frac{d}{dt}$. [15 marks]

Use the above result to find the general solution of the following differential equation

$$x^3 y''' + xy' - y = 3x^4,$$

where $' \equiv \frac{d}{dx}$. [30 marks]

(b) With $D \equiv \frac{d}{dx}$, solve the following simultaneous differential equations.

$$\begin{aligned} (D+1)x + (2D+7)y &= e^t + 2, \\ -2x + (D+3)y &= e^t - 1. \end{aligned}$$

[35 marks]

(c) Show that the orthogonal trajectories of the equation

$$x^2 + (y-c)^2 = 1 + c^2,$$

where c is a constant, are the circles

$$(x + c^*)^2 + y^2 = c^{*2} - 1,$$

where c^* is a constant. [20 marks]

Q4. (a) Let the general second-order differential equation

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_o(x)y = 0$$

have two linearly independent solutions $y_1(x)$ and $y_2(x)$ such that

$$y_1(x) = |x|^{r_1}\left(1 + \sum_{k=1}^{\infty} c_k x^k\right),$$

where r_1 and r_2 are the solutions of the indicial equation with $\mathrm{Re}(r_1) \geq \mathrm{Re}(r_2)$. Assume that all other conditions are satisfied. Write down the forms of $y_2(x)$ by categorizing the type of r_1 and r_2. [30 marks]

(b) Let the linear second-order ordinary differential equation be

$$x(x-1)y'' + (3x-1)y' + y = 0. \quad (1)$$

(i) Find the roots of the indicial equation of the given differential equation (1). [20 marks]

(ii) If

$$y_1(x) = \sum_{m=0}^{\infty} x^m = \frac{1}{1-x}, \qquad |x| < 1,$$

is one of the solution of (1), write down the second independent solution $y_2(x)$ of (1) in terms of $y_1(x)$. [10 marks]

(iii) Find the general solution of (1) in series form. [40 marks]

Q5. (a) By identifying the type of the following differential equation find its general solution.

$$y = xy' - \frac{y'}{\sqrt{1 + y'^2}}.$$

[20 marks]

(b) Write down the condition of integrability and of exactness of the total differential equation

$$P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz = 0.$$

Hence solve the following differential equations.

$$yz\,dx + (zx - yz^3)dy - 2xy\,dz = 0.$$

$$(x - y)dx - x\,dy + z\,dz = 0.$$

[35 marks]

(c) (i) Find the general solution of the partial differential equation

$$q(1 - p^2) = p(1 - z)$$

and discuss about its singular solution, here $p = \dfrac{\partial z}{\partial x}$ and $q = \dfrac{\partial z}{\partial y}$.

[20 marks]

(ii) Apply Charpit's method to find the complete and singular solutions of the nonlinear first-order partial differential equation

$$p(q^2 + 1) + (b - z)q = 0.$$

[25 marks]

Q6. (a) Show that the fourier series of the function

$$f(x) = \begin{cases} -k, & \text{if } -\pi < x < 0, \\ k, & \text{if } \quad 0 < x < \pi, \end{cases} \qquad \text{and} \quad f(x + 2\pi) = f(x)$$

is

$$\frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right).$$

Hence show that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

[40 marks]

(b) Use fourier transform to show that the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where c is a constant, subject to the initial condition

$$u(x, 0) = f(x), \qquad -\infty < x < \infty,$$

where $f(x)$ is the given initial temperature, gives the solution

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\lambda) \exp(-c^2\lambda^2 t) \exp(i\lambda x)\ d\lambda,$$

where

$$F(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(v) \exp(-iv\lambda) \, dv$$

is the fourier transform. [30 marks]

Hence prove that the solution $u(x,t)$, using convolution equation, can be written as

$$u(x,t) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{+\infty} f(p) \exp\left\{-\frac{(x-p)^2}{4c^2 t}\right\} dp.$$

You may use the following definition of convolution.

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(p) g(x-p) \, dp.$$

[30 marks]