EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE (1996/97)

(June/August'2004)

EXTERNAL DEGREE

EXMT 201 - VECTOR SPACES AND MATRICES

Answer only four questions Time: Two hours

Q1. (a) Define what is meant by

(i) a vector space;

(ii) subspace of a vector space.

Let $\mathbb{Q}(\sqrt{2}) = \{x + y\sqrt{2} : x, y \in \mathbb{Q}\}$. The operations $\oplus, \odot$ on $\mathbb{Q}(\sqrt{2})$ are defined as follows:

$$(x_1 + y_1\sqrt{2}) \oplus (x_2 + y_2\sqrt{2}) = (x_1 + x_2) + (y_1 + y_2)\sqrt{2},$$

$$\alpha \odot (x_1 + y_1\sqrt{2}) = \alpha x_1 + \alpha y_1\sqrt{2}, \quad \forall\ \alpha, x_1, x_2, y_1, y_2 \in \mathbb{Q}.$$

Show that $\left(\mathbb{Q}(\sqrt{2}), \oplus, \odot\right)$ forms a vector space over $\mathbb{Q}$.

(b) Let W_1 and W_2 be two subspaces of a vector space V over a field F and let A_1 and A_2 be non-empty subsets of V. Show that

(i) $W_1 + W_2$ is the smallest subspace containing both W_1 and W_2;

(ii) if A_1 spans W_1 and A_2 spans W_2 then $A_1 \cup A_2$ spans $W_1 + W_2$.

(c) Which of the following sets are subspaces of $\mathbb{R}^3$? In each case justify your answer.

(i) $W_1 = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1\}$

(ii) $W_2 = \{(x, y, z) \in \mathbb{R}^3 : x + y^2 = 0\}$.

Q2. (a) Define the following:

(i) A linearly independent set of vectors;

(ii) A basis for a vector space;

(iii) Dimension of a vector space.

(b) Let V be an $n-$ dimensional vector space.

Prove the following:

(i) A linearly independent set of vectors of V with n elements is a basis for V;

(ii) Any linearly independent set of vectors of V may be extended as a basis for V;

(iii) If L is a subspace of V, then there exists a subspace M of V such that $V = L \oplus M$.

(c) Extend the subset $\{(1, -2, 5, -3), (0, 7, -9, 2)\}$ to a basis for $\mathbb{R}^4$.

Q3. (a) Define

(i) Range space $R(T)$;

(ii) Null space $N(T)$

of a linear transformation T from a vector space V into another vector space W.

Find $R(T)$, $N(T)$ of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by

$$T(x, y, z) = (x + 2y + 3z, x - y + z, x + 5y + 5z) \ \ \forall \ (x, y, z) \in \mathbb{R}^3.$$

Verify the equation $\dim V = \dim(R(T)) + \dim(N(T))$ for this linear transformation.

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $T(x, y, z) = (x,\ x + y,\ y - z)$ and let $B_1 = \{(1,0,0), (0,1,0), (0,0,1)\}$ and $B_2 = \{(1,1,0), (-1,1,0), (0,0,1)\}$ be bases for $\mathbb{R}^3$.

Find

(i) The matrix representation of T with respect to the basis B_1;

(ii) The matrix representation of T with respect to the basis B_2 by using the transition matrix;

(iii) The matrix representation of T with respect to the basis B_2 directly.

Q4. (a) Define the following terms

(i) Rank of a matrix;

(ii) Echelon form of a matrix;

(iii) Row reduced echelon form of a matrix.

(b) Let A be an $m \times n$ matrix. Prove that

(i) Row rank of A is equal to column rank of A;

(ii) If B is an $m \times n$ matrix obtained by performing an elementary row operation on A, then $r(A) = r(B)$.

(c) Find the rank of the matrix

$$\begin{pmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{pmatrix}.$$

(d) Find the row reduced echelon form of the matrix

$$\begin{pmatrix} -1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{pmatrix}.$$

Q5. (a) Define the following terms as applied to an $n \times n$ matrix $A = (a_{ij})$.

(i) Cofactor A_{ij} of an element a_{ij};

(ii) Adjoint of A.

Prove that

$$A \cdot (adj\, A) = (adj\, A) \cdot A = det\, A \cdot I$$

where I is the $n \times n$ identity matrix.

(b) If A and B are two $n \times n$ non-singular matrices, then prove that

(i) $adj(\alpha A) = \alpha^{n-1} \cdot adj\, A$ for every real number α,

(ii) $adj(AB) = (adj\, B)\,(adj\, A)$;

(iii) $adj(A^{-1}) = (adj\, A)^{-1}$;

(iv) $adj(adj\, A) = (det\, A)^{n-2} A$;

(v) $adj(adj(adj\, A)) = (det\, A)^{n^2-3n+3} A^{-1}$.

(c) Find the inverse of the matrix

$$\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}.$$

Q6. (a) State the necessary and sufficient condition for a system of linear equations to be consistent.

Reduce the augmented matrix of the following system of linear equations to its row reduced echelon form and hence determine the values of a such that the system has

(i) A unique solution;

(ii) No solution;

(iii) More than one solution.

$$\begin{aligned} x + y - z &= 1 \\ 2x + 3y + az &= 3 \\ x + ay + 3z &= 2. \end{aligned}$$

(b) State **Crammer's** rule for 3×3 matrix and use it to solve

$$\begin{aligned} x + 2y + 3z &= 10 \\ 2x - 3y + z &= 1 \\ 3x + y - 2z &= 9. \end{aligned}$$

(c) Prove that the system,

$$\begin{aligned} x + 2y - 3z &= -1 \\ 3x - y + 2z &= 7 \\ 5x + 3y - 4z &= 2 \end{aligned}$$

is inconsistent.