EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE 1996/97

(June/July' 2004)

EXTERNAL DEGREE

EXMT 202 - METRIC SPACE & RIEMANN INTEGRAL

Answer four questions only

Time : Two hours

1. Define the term "metric space".

(a) Let $X = C_{[0,1]}$ be the set of all continuous real valued functions on $[0, 1]$, and let

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$$

for $f, g \in X$. Prove that (X, d) is a metric space.

(b) Let (Y, d) be a subspace of a metric space (X, d). Prove that $A \subseteq Y$ is an open set in (Y, d) if and only if there exists an open set G in (X, d) such that $A = Y \cap G$.

(c) Prove that every Cauchy sequence in a metric space is bounded.

2. Let A be a subset of a metric space (X, d). Define the following terms.

- Interior point of A,
- Interior of A,
- Closure of A.

(a) Prove that every point of an open ball is an interior point.

(b) Prove that, for any subset A of a metric space its closure $(\overline{A})$ is the smallest closed set containing A.

(c) Let A be a non empty subset of a metric space and let $r > 0$. Prove that $a \in \overline{A}$ if and only if $B(a, r) \cap A \neq \phi$ for every open ball $B(a, r)$, where $\overline{A}$ denotes the closure of A.

(d) Is it true that, arbitrary intersection of open sets is open? Justify your answer.

3. Define the following terms in a metric space.

- Separated sets,
- Disconnected set.

(a) Prove that, a metric space (X, d) is disconnected if and only if there exists a non empty subset of X which is both open and closed.

(b) Prove that if two open sets A and B are separated in a metric space then A and B are disjoint sets.

(c) Prove that a metric space (X, d) is disconnected if and only if it can be written as the union of two non-empty disjoint open sets.

4. Define the term "compact set" in a metric space.

   (a) Let A be a compact subset of a metric space X and let $p \in X \setminus A$. Prove that there exists open sets G and H such that $p \in G$, $A \subseteq H$ and $G \cap H = \Phi$.

   (b) Prove that every compact subset of a metric space is closed and bounded.

   (c) Prove that the continuous image of a compact set is compact.

5. Let f be a real valued bounded function on $[a, b]$. Explain what is meant by the statement that "f is Riemann integrable over $[a, b]$".

   (a) With the usual notations, prove that a bounded function f on $[a, b]$ is Riemann integrable if and only if for given $\epsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \epsilon.$$

   (b) Prove that if f is continuous on $[a, b]$, then

      i. f is Riemann integrable over $[a, b]$.

      ii. the function $F : [a, b] \longrightarrow \mathbb{R}$ defined by $F(x) = \int_a^x f(t)\, dt$ is differentiable on $[a, b]$ and $F'(x) = f(x) \quad \forall\, x \in [a, b]$.

6. When is an integral $\int_a^b f(x)\,dx$ said to be an improper integral of the first kind, the second kind and the third kind?

What is meant by the statement " an improper integral of the second kind is convergent"?

Discuss the convergence of the improper integral $\int_a^b \frac{dx}{(x-a)^p}$, where a and b are real numbers.

Test the convergence of the following:

(a) $\int_0^1 \frac{e^x}{\sqrt{x}}\,dx$;

(b) $\int_3^6 \frac{\ln x}{(x-3)}\,dx$;

(c) $\int_0^1 \frac{dx}{\sqrt{x}\sqrt{1+4x^2}}\,dx$.