EASTERN UNIVERSITY, SRI LANKA

FIRST EXAMINATION IN SCIENCE 1996/97

(June/July' 2004) (Repeat)

EXTERNAL DEGREE

EXMT 103 & 104 - VECTOR ALGEBRA & CLASSICAL MECHANICS I

Answer four questions only selecting two from each section

Time : Two hours

Section A

1. (a) For any three vectors $\underline{a}$, $\underline{b}$, $\underline{c}$, prove the identity

$$\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \cdot \underline{c})\, \underline{b} - (\underline{a} \cdot \underline{b})\, \underline{c} .$$

Hence prove that

$$(\underline{a} \wedge \underline{b}) \cdot (\underline{c} \wedge \underline{d}) = (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d}) - (\underline{a} \cdot \underline{d})(\underline{b} \cdot \underline{c}) .$$

(b) Let $\underline{l}$, $\underline{m}$ and $\underline{n}$ be three non-zero and non-coplanar vectors such that any two of them are not parallel. By Considering the vector product $(\underline{r} \wedge \underline{l}) \wedge (\underline{m} \wedge \underline{n})$, prove that any vector $\underline{r}$ can be expressed in the form

$$\underline{r} = (\underline{r} \cdot \underline{\alpha})\, \underline{l} + (\underline{r} \cdot \underline{\beta})\, \underline{m} + (\underline{r} \cdot \underline{\gamma})\, \underline{n} .$$

Find the vectors $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$ in terms of $\underline{l}$, $\underline{m}$, $\underline{n}$.

(c) A vector $\underline{r}$ satisfies the equation

$$\underline{r} \wedge \underline{b} = \underline{c} \wedge \underline{b} \text{ and } \underline{r} \cdot \underline{a} = 0 ,$$

where $\underline{a}$ and $\underline{b}$ are non-zero and not perpendicular vectors. Show that $\underline{r}$ can be expressed in the form

$$\underline{r} = \underline{c} - \lambda \underline{b} ,$$

where λ is a scalar.

2. (a) Define the following terms.

i. The gradient of a scalar field ϕ,

ii. The divergence of a vector field $\underline{F}$,

iii. The curl of a vector field $\underline{F}$.

(b) Prove the following:

i. $\text{div}(\phi\, \underline{F}) = \phi\, \text{div}\underline{F} + \text{grad}\phi \cdot \underline{F}$,

ii. $\text{curl}\,(\phi\, \underline{F}) = \phi\, \text{curl}\underline{F} + \text{grad}\phi \wedge \underline{F}$.

(c) Let $\underline{a}$ be non-zero constant vector and $\underline{r}$ be a position vector of a point such that $\underline{a} \cdot \underline{r} \neq 0$ and let n be a constant. If $\phi = (\underline{a} \cdot \underline{r})^n$, show that $\nabla^2 \phi = 0$ if and only if $n = 0$ or $n = 1$.

If $r = |\, \underline{r} \,|$, find $\text{div}(r^n \underline{r})$ and $\underline{\nabla} \left(\frac{\underline{a} \cdot \underline{r}}{r^5} \right)$.

Hence show that

$$\text{curl} \left[\left(\frac{\underline{a} \cdot \underline{r}}{r^5} \right) \underline{r} \right] = \frac{\underline{a} \wedge \underline{r}}{r^5}.$$

3. (a) Define the terms "Conservative vector field" and "Solenoidal vector field".

Show that

$$\underline{F} = (2x - y)\underline{i} + (2yz^2 - x)\underline{j} + (2y^2z - z)\underline{k}$$

is conservative but not solenoidal.

(b) State and prove Green's theorem.

Evaluate $\oint_C (xy + y^2)dx + x^2dy$ where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

Section B

4. Prove that the radial and transverse component of the acceleration of a particle in a plane in terms of polar co-ordinates (r, θ) are

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \quad \text{and} \quad \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

respectively.

A particle on a smooth table is attached to a string passing through a small hole in the table and carries an equal particle hanging vertically. The former particle is projected along the table at right angle to the string with velocity $\sqrt{2gh}$ when at a distance 'a' from the hole. If r is the distance of the former particle from the hole at time t, prove the following results:

(a) $\left(\frac{dr}{dt}\right)^2 = gh\left(1 - \frac{a^2}{r^2}\right) + g(a - r)$;

(b) The lower particle will be pulled up to the hole if $2h > a$ and the total length of the string is less than $\frac{h}{2} + \sqrt{ah + \frac{h^2}{4}}$;

(c) Tension of the string is $\frac{1}{2}mg\left(1 + \frac{2a^2h}{r^3}\right)$, where m is the mass of each particle.

5. A particle moves in a plane with velocity v and the tangent to the path of the particle makes an angle ψ with a fixed line in the plane. Prove that the component of acceleration of the particle along the tangent and perpendicular to it are $\frac{dv}{dt}$ and $v\frac{d\psi}{dt}$ respectively.

A body attached to a parachute is released from an aeroplane which is moving horizontally with velocity v_0. The parachute exerts a drag opposing motion which is k times the weight of the body, where k is a constant. Neglecting the air resistance to the motion of the body, prove that if v is the velocity of the body when its path is inclined an angle ψ to the horizontal, then

$$v = \frac{v_0 \sec\psi}{(\sec\psi + \tan\psi)^k}.$$

Prove that if $k = 1$, the body cannot have a vertical component of velocity greater than $\frac{v_0}{2}$.

6. State the angular momentum principle for motion of a particle.

A right circular cone with a semi vertical angle α is fixed with its axis vertical and vertex downwards. A particle P of mass m is held at the point A on the smooth inner surface of the cone at a distance 'a' from the axis. If it is a given velocity 'u' in the horizontal direction perpendicular to OA, where O is the vertex of the cone, through out of the motion the path of the particle is inner surface of the cone. Show that the particle rises above the level of A if $u^2 > ag\cot\alpha$ and greatest reaction between the particle and the surface is

$$mg\left(\sin\alpha + \frac{u^2}{ag}\cos\alpha\right).$$